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An Inverse Problem In Convex Optimization.

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Dedication

Thankfully I dedicate this thesis to all those who contributed to its success. I dedicate it to my beloved family: father, mother, brother and sisters who through their encouragement, patience and support enabled me to continue my study and get this degree.

I am also very grateful to my university teachers and professors who were like a candle to me and that enlightened my mind and life.

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Abstract

This research aims mainly to solve an inverse problem arising in convex optimization.

$$(\mathcal{P})\left\{\ \max_x f(x) \quad ; \quad Ax = C(A),\right.$$

where f is a strictly increasing funnction with respect to each coordinate of the vector x, the Hessian matrix $D_x^2 f$ is negative definite on the subspace $\{D_x f\}^{\perp}$, f is of class C^2, A is an $m \times n$ matrix of rank m, $C : \mathbb{R}_{++}^{m\times n} \to \mathbb{R}_{++}^m$ is homogeneous of degree one and $x \in \mathbb{R}^n$.

We consider a maximization problem under m linear constraints, we characterize the solutions of this kind of problems and give necessary and sufficient conditions for a given function in $\mathbb{R}^n$ to be the solution of a multi-constraints maximization problem.

CONTENTS

1. INTRODUCTION

Some inverse problems arise in microeconomic theory, in which we are required to characterize the solution of some optimization problems, under one or many linear constraints. The solution of the optimization problem is called the individual demand function.

In the standard individual problem, the individual maximizes a function that represents his tastes, called the utility function, under his budget constraints.
The individual demand function is fully characterized by the well known conditions (i) homogeneity of degree zero, (ii) Walras Law, and (iii) symmetry and negative semi-definiteness of its substitution matrix.

The individual demand is the solution to the utility maximization problem under the budget constraint $p^T x = w$, where p is the price vector, and w is the individual income, where p^T is the transpose of p.

The standard utility maximization problem under the budget constraint takes the form

$$\mathcal{P}\left\{\max_x U(x) \quad ; \quad p^T x = w(p)\right.$$

where U is utility function that satisfies certain smoothness, monotonicity, and concavity conditions, and $w(p)$ is convex and homogeneous of degree one. The solutions of this problem are characterized in [2].

In the general case, a multi-constraints optimization problem takes the form:

$$\mathcal{P}\left\{\max_x f(x) \quad ; \quad Ax \leq C(A),\right.$$

where $x \in \mathbb{R}^n$, A is $m \times n$ matrix and f and C are some functions. Hence, we are dealing with a multi-constraints maximization problem with linear constraints. The solution of this problem is a function of the parameters $A = (a^i_j)$, $i, j = 1, ...m$. We assume certain conditions on the functions f and C that guarantee the differentiability of the solutions which we require to be at least of class C^2. Our main objective is to characterize the solutions of this type of optimization problems. We rely on the first order conditions and optimality conditions to achieve our objective. Moreover, we make use of the envelope theorem and the value function, $V(A) = f(x(A))$, of the above problem. The inverse problem arising in this case was addressed in [3].

Such kind of problems arise in many applications especially in some economic contexts in microeconomic theory. Economic applications to this problem will be given in the sequel. Moreover, we will show that the results we get here generalize well-known results in consumer theory, see [7] for a recent survey. An inverse problem arising from economic theory was also solved by Ekeland and Djitt'e [8]. We use the indirect approach to deal with this problem. This approach depends on the value function, $V(A)$.

The necessary and sufficient conditions on a given function $x(A) \in \mathbb{R}^n$ for the existence of a value function will be given. It turns out that the necessary and sufficient conditions will include a set of function λ_{ij} , $i, j = 1, ..., m$ that can be computed from $x(A)$. The problem then is to find the objective function. This is a duality problem.

Our problem will be split into mathematical integration problem and economic integration problem.

- **Mathematical integration.** Given a function $x(A)$ and a family of functions λ_{ik}, $1 \leq i, k \leq m$, what are the necessary and sufficient conditions for the existence of $m+1$ functions $\lambda_1, ..., \lambda_m$ and V that satisfy equation
$$\frac{1}{\lambda_k}\frac{\partial V}{\partial a^i_j} = \lambda_{ik}\left(\frac{\partial C^i}{\partial a^i_j} - x^j\right) \tag{1.1.1}$$
with $\lambda_{ik} = \frac{\lambda_i}{\lambda_k}$ and $C^i(a^i) = (a^i)^T x(A)$.

- **Economic integration.** In addition to the mathematical integration, we impose the following additional conditions on the functions that satisfy (1.1.1); the functions λ_i are strictly positive and the function V is quasi-convex with respect to each a^i for all $i = 1, ..., m$.

To get the necessary and sufficient conditions for mathematical integration, we use the techniques of exterior differential calculus that showed to be powerful for the treatment of such problems. A good reference to these techniques is the book by Bryant et al. [6]. We get local results; that is, the functions involved in the integration problem are defined in a neighbourhood of some given point. We define a family of differential forms and set up an integration problem using these forms. The solution of this integration problem, then, requires solving a nonlinear system of partial differential equations. The integration problem will be solved using Darboux Theorem [6].

This thesis consists mainly of 5 chapters, where chapter 2 consists of basic definitions and results, and chapter 3 reviews exterior differential calculus concepts. In chapter 4 solution of single constraint and non-homogeneous models are given. Chapter 5 contains the our main results. The main results that include the necessary and sufficient conditions for mathematical integration are given. Then, the economic integration problem is solved. Finally, duality problem is considered in section 6 of this chapter.

2. BASIC DEFINITIONS AND RESULTS

In this section, we state some basic definitions and results that we need in this thesis.

Definition 1 (Homogeneous Function)**.**
Let D be a subset of $\mathbb{R}^n$, $f : D \to \mathbb{R}$ be a C^1 function defined on D. Then, f is said to be homogenous of degree $k \in \mathbb{R}$, if for any real number $t > 0$, the following condition holds

$$f(tx^1, ..., tx^n) = t^k f(x^1, ..., x^n), \quad \forall\ x \in D.$$

Theorem 2.1 (Euler's Theorem)**.**
Let D be a subset of $\mathbb{R}^n$, $f : D \to \mathbb{R}$ be a C^1 function defined on D, f is k-homogenous if and only if

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x)x^i = kf(x^1, ..., x^n), \quad \forall\ x \in D.$$

Definition 2 (Convex Set)**.**
A set $D \subseteq \mathbb{R}^n$ is called convex if for any $x, y \in D$, $\lambda \in (0, 1)$, the element

$$\lambda x + (1 - \lambda)y \in D.$$

Definition 3 (Convex Function)**.**
Let $D \subseteq \mathbb{R}^n$ be a convex set. Then, f is convex function on D if

$$f[\lambda x_1 + (1 - \lambda)x_2] \leq \lambda f(x_1) + (1 - \lambda)f(x_2)$$

for all $\lambda \in [0, 1]$ and all $x_1, x_2 \ \in D$.

Definition 4 (Quasi-convex Function).
Let D be a subset of $\mathbb{R}^n$, consider a function $f : D \to \mathbb{R}$ where D is convex set. Then f is quasi-convex on D if

$$f[\lambda x_1 + (1-\lambda)x_2] \leq \max\{f(x_1), f(x_2)\}$$

for all $\lambda \in [0,1]$ and all $x_1, x_2 \in D$.

Theorem 2.2 (Envelope Theorem for Constrained Problems).
Let $x^*(a) = (x_1^*(a), ..., x_n^*(a))$ denote the solution to the following problem:

$$\max \quad f(x;a)$$

$$s.t.\ g_1(x;a) = 0,\ ..., g_k(x;a) = 0$$

Let $\lambda_1(a), ..., \lambda_k(a)$ be the lagrange multipliers for each constraint in this problem. Then $\underbrace{\frac{d}{da} f(x^*(a), a)}_{\text{Total derivative of the original function } f} = \underbrace{\frac{\partial}{\partial a} L(x^*(a), \lambda(a), a)}_{\text{Partial derivative of Lagrangian}}$.

Theorem 2.3 (Implicit Function Theorem).
Let $X \times P$ be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, and let $f : X \times P \to \mathbb{R}^n$ be C^k, for $k \geq 1$. Assume that $D_x f(\bar{x}, \bar{p})$ is invertible. Let $\bar{y} = f(\bar{x}, \bar{p})$, then there are neighborhoods $\mathbf{U} \subset X$ and $\mathbf{W} \subset P$ of $\bar{x}$ and $\bar{p}$ on which the equation $f(x,p) = y$ uniquely defines x as a function of p. That is, there is a function $\xi : \mathbf{W} \to \mathbf{U}$ such that:

(a) $f(\xi(p); p) = \bar{y}$ for all $p \in \mathbf{W}$.

(b) For each $p \in \mathbf{W}$, $\xi(p)$ is the unique solution to $f(x,p) = y$ lying in $\mathbf{U}$. In particular, then $\xi(\bar{p}) = \bar{x}$.

(c) ξ is C^k on $\mathbf{W}$.

Definition 5 (Positive Definite Matrices).
A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is called positive definite if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$.

Definition 6 (Positive Semidefinite Matrices)**.**
A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called positive semidefinite if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$ and there exists an $x \neq \overrightarrow{0}$ such that $x^T Ax = 0$.

Theorem 2.4. *[11] Let D be an open convex subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a C^2 function. Then, f is convex if and only if $D^2 f(x)$ is a positive semidefinite matrix for all $x \in D$.*

Theorem 2.5. *[11] Let D be a convex subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a function. Then, if f is convex on D, then it is also quasi-convex on D.*

Theorem 2.6. *[11] Let $f : D \to \mathbb{R}$ be a C^2 function defined on an open convex set D with everywhere nonzero first partial derivatives. Then, f is quasi-convex if and only if for all $x \in D$,*

$$y'H(x)y \geq 0 \quad \textit{whenever} \quad \nabla f'(x)y = 0$$

where $H(x)$ and $\nabla f(x)$ are respectively the Hessian matrix, and the gradient of the function $f(x)$.

Theorem 2.7. *[11] Suppose that $f(x)$ is twice differentiable at $\bar{x}$. If $\nabla f(\bar{x}) = 0$ and $H(\bar{x})$ is positive definite, then $\bar{x}$ is a local minimum.*

3. EXTERIOR DIFFERENTIAL CALCULUS

3.1 Introduction

Exterior differential calculus is a mathematical tool which was developed in the early twentieth century to solve problems in group theory and geometry, but it recently turned out to be extremely useful for solving problems in the economic theory of demands.

There are two major operations: a purely algebraic one, the exterior product (also called the wedge product), and denoted by $\wedge$, and special kind of differentiation, called the exterior derivative, and is denoted by d. They operate on differential forms, which are classified by their degrees: differential forms of degree 0 are just functions, differential forms of degree 1 are analogous to vector fields, and differential forms of degree $k \geq 1$ arise from differential forms of lower degree by taking exterior products and/or exterior derivatives. All the machinery of differential forms and exterior differential calculus is directed towards proving and applying two major theorems: the Darboux theorem and the Cartan-Kähler theorem.

3.2 Differential Manifolds

Our first goal is to define the notion of a manifold. Manifolds are, roughly speaking, abstract surfaces that locally look like linear spaces. We shall assume at first that the linear spaces are R^n for a fixed integer n, which will be the dimension of the manifold.

3.2.1 Manifolds and Atlases

Definition 7. *[10][Manifold]*
A manifold M of dimension n is a topological space M, such that every point $x \in M$ has a neighbourhood which is homeomorphic to an open set in Euclidean space R^n.

Definition 8. *[10] [Chart]*
A chart for M is a homeomorphism $\phi : U \to V$ where U is open in M and V is open in R^n.

Definition 9. *[10][Atlas]*
A collection of charts $\mathcal{A} = \{\varphi_\alpha : U_\alpha \to V_\alpha | \alpha \in \mathcal{I}\}$ is called an atlas for M if $\bigcup_{\alpha \in \mathcal{I}} = M$.

Example 3.1. *[10]*
$\mathbb{R}^n$ or any open subset of $\mathbb{R}^n$ is a smooth manifold with an atlas consisting of one chart. The unit sphere

$$S^n = \{(x^0, x^1, ..., x^n) | \sum_{i=0}^{n} (x^i)^2 = 1\}$$

has an atlas consisting of two charts $(U_\pm, \phi_\pm)$, where $U_\pm = S^n \backslash \{(\pm 1, 0, ..., 0)\}$ and

$$\phi_\pm(x^0, x^1, ..., x^n) = \frac{1}{\pm 1 - x^0}(x^1, ..., x^n)$$

Definition 10. *[10] Two differentiable atlases $\mathcal{A}$ and $\mathcal{B}$ are compatible if their union is also a differentiable atlas. Equivalently, for every chart ϕ in $\mathcal{A}$ and η in $\mathcal{B}$, ϕ o η^{-1} and η o ϕ^{-1} are smooth.*

Definition 11. *[10][Smooth Manifold]*
A smooth manifold is a set of points together with a finite set of subsets $U_\alpha \subset M$ and one to one mappings

$$\phi_\alpha : U_\alpha \to \mathbb{R}^n$$

such that

1. $\cup_\alpha U_\alpha = M$.

2. *For every nonempty intersection $U_\alpha \cap U_\beta$, the set $\phi_\alpha(U_\alpha \cap U_\beta)$ is an open subset of $\mathbb{R}^n$ and the one to one mapping $\phi_\beta \circ \phi_\alpha^{-1}$ is a smooth function on $\phi_\alpha(U_\alpha \cap U_\beta)$.*

Definition 12. *[5][Smooth Function]*
A function f on $\mathcal{M}$ into $\mathcal{N}$ is said to be smooth if for every $p \in \mathcal{U}$ there is a chart $(\mathcal{U}, \phi)$ for $\mathcal{M}$ and a chart $(\mathcal{V}, \psi)$ for $\mathcal{N}$ at $f(p)$ with $f(\mathcal{U}) \subseteq \mathcal{V}$ such that the partial derivatives of

$$\psi \ o \ f \ o \ \phi^{-1} : \phi(\mathcal{U}) \subseteq \mathbb{R}^m \to \psi(\mathcal{V}) \subseteq \mathbb{R}^n$$

exist and are continuous to all orders, i.e, $\psi \ o \ f \ o \ \phi^{-1}$ is smooth.

Definition 13. *[10] A differentiable structure on a manifold M is an equivalence class of differentiable atlases, where two atlases are deemed equivalent if they are compatible.*

3.2.2 Tangent Vectors

Two curves $t \to c_1(t)$ and $t \to c_2(t)$ in an n-manifold M are called equivalent at the point m if

$$c_1(0) = c_2(0) = m \ \ and \ \ (\varphi \ o \ c_1)'(0) = (\varphi \ o \ c_2)'(0)$$

in some chart φ.

Definition 14. *[9] Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and let $\mathbf{v}$ be a vector in $\mathbb{R}^n$. We define the directional derivative in the $\mathbf{v}$ direction at a point $x \in \mathbb{R}^n$ by*

$$D_{\mathbf{v}} f(x) = \frac{d}{dt} f(x + t\mathbf{v}) \mid_{t=0} = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i}(x).$$

The tangent vector at the point x may then be defined as

$$\mathbf{v}(f(x)) \equiv D_{\mathbf{v}}(f(x))$$

Definition 15. *[9][Tangent Vectors]*
Let M be a differentiable manifold and let $A(M)$ be the algebra of real-valued differentiable functions M. Then the tangent vector to M at a point x in the manifold is given by the derivation $D_v : A(M) \to \mathbb{R}$.

Definition 16. *[10][Tangent Space]*
A tangent space to M at $m \in M$ is the set of tangent vectors to M at m which forms a vector space, and it is denoted by $T_m M$.

Definition 17. *[10][Tangent Bundles]*
The tangent bundle of M, denoted by TM, is the set that is the disjoint union of the tangent spaces to M at the points $m \in M$, that is,

$$TM = \cup_{m \in M} T_m M.$$

Thus, a point of TM is a vector v that is tangent to M at some point $m \in M$. If M is an n-manifold, then TM is a $2n$-manifold.

3.2.3 Differentials and Covectors

If $f : M \to \mathbb{R}$ is a smooth function, we can differentiate it at any point $m \in M$ to obtain a map $T_m f : T_m M \to T_{f(m)}\mathbb{R}$. Identifying the tangent space of $\mathbb{R}$ at any point with itself, we get a linear map covector $df(m) : T_m M \to \mathbb{R}$. That is, $df(m) \in T_m^* M$, the dual of the vector space $T_m M$. We call df the differential of f.

For $v \in T_mM$, we call $df(m).v$ the directional derivative of f in the direction v. We now identify a basis of T_mM using the operators $\frac{\partial}{\partial x_i}$. We write

$$\{e_1, ..., e_n\} = \left\{ \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right\}$$

for this basis, so that $v = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x_i}$.

If we replace each vector space T_mM with its dual T_m^*M, we obtain a new $2n$-manifold called the cotangent bundle and denoted by T_m^*M. The dual basis to $\frac{\partial}{\partial x_i}$ is denoted by dx_i. Thus, relative to a choice of local coordinates we get the basic formula

$$df(x) = \sum \frac{\partial f}{\partial x_i} dx_i$$

for any smooth function $f : M \to \mathbb{R}$.

Definition 18. *[10][Multilinear map]*
A map $\alpha : V \times ... \times V$ (there are k-factors) $\to \mathbb{R}$ is multilinear when it is linear in each of its factors, that is,

$$\alpha(v_1, ..., av_j + bv'_j, ..., v_k) = a\alpha(v_1, ..., v_j, ..., v_k) + b\alpha(v_1, ..., v'_j, ..., v_k)$$

for all j with $1 \leq j \leq k$.

Definition 19. *[5][Tensor]*
A tensor of type (k, l) at x is a multilinear map which takes k vectors and l covectors and gives a real number

$$T_x : \underbrace{T_xM \times ... \times T_xM}_{k \ times} \times \underbrace{T_x^*M \times ... \times T_x^*M}_{l \ times} \to \mathbb{R}.$$

Note that a covector is just a tensor of type $(1, 0)$, and a vector is a tensor of type $(0, 1)$, since a vector v acts linearly on a covector ω by $v(\omega) := \omega(v)$.

3.3 Differential Forms

The main idea of differential forms is to provide a generalization of the basic operations of vector calculus, div, grad, and curl, and the integral theorems of Green, Gauss, and Stokes to manifolds of arbitrary dimension.

We have already met one-forms, a term that is used in two ways. They are either members of a particular cotangent space T_m^*M or else, analogous to a vector field, an assignment of a covector in T_m^*M to each $m \in M$. A basic example of a one-form is the differential of a real-valued function.

Definition 20. *[10][Differential 1-form]*
A 1*-form* α *on a manifold* M *is a linear smooth function* $\alpha(m) : T_mM \to \mathbb{R}$ *on tangent vectors.*

Definition 21. *[10][Differential 2-form]*
A 2*-form* α *on a manifold* M *is a function* $\alpha(m) : T_mM \times T_mM \to \mathbb{R}$ *that assigns to each point* $m \in M$ *a skew-symmetric bilinear form on the tangent space* T_mM *to* M *at* m*.*

Definition 22. *[10][Differential k-form]*
A k*-form* α *on a manifold* M *is a function* $\alpha(m) : T_mM \times ... \times T_mM$ *(there are* k *factors)*$\to \mathbb{R}$ *that assigns to each point* $m \in M$ *a skew-symmetric* k*-multilinear map on the tangent space* T_mM *to* M *at* m*.*

A k^{th} order differential form $(k - form)$ on $\mathbb{R}^n$ is a sum of terms of the form

$$f(x)dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}.$$

subject to the rule

$$dx_i \wedge dx_j = -dx_j \wedge dx_i.$$

Definition 23. *[10][$(0,k)$-tensor]*
Differential form of degree k without the skew-symmetry assumption

Definition 24. *[10][Skew map]*
A k-multilinear map $\alpha : V \times ... \times V \to \mathbb{R}$ is skew-symmetric when it changes sign whenever two of its arguments are interchanged, that is, for all $v_1, ..., v_k \in V$,

$$\alpha(v_1, ..., v_i, ..., v_j, ..., v_k) = -\alpha(v_1, ..., v_j, ..., v_i, ..., v_k).$$

3.4 Tensor and Wedge Products

Definition 25. *[5][Tensor Product]*
Let T and S be two tensors at x of types (k,l) and (p,q) respectively. Then the tensor product $T \otimes S$ is the tensor at x of type $(k+p, l+q)$ defined by

$$\begin{aligned} T \otimes S(v_1, ..., v_{k+p}, w_1, ..., w_{l+q}) &= T(v_1, ..., v_k, w_1, ..., w_l) \\ &\times S(v_{k+1}, ..., v_{k+p}, w_{l+1}, ..., w_{l+q}) \end{aligned}$$

for all vectors $v_1, ..., v_{k+p} \in T_xM$ and all covectors $w_1, ..., w_{l+q} \in T_x^*M$.

Definition 26. *[5] If α is a $(0,k)$-tensor on a manifold M and β is a $(0,l)$- tensor, their tensor product $\alpha \otimes \beta$ is the $(0, k+l)$-tensor on M defined by*

$$(\alpha \otimes \beta)_m(v_1, ..., v_{k+l}) = \alpha_m(v_1, ..., v_k)\beta_m(v_{k+1}, ..., v_{k+l})$$

at each point $m \in M$.

Definition 27. *[10][Alternation Operator $\boldsymbol{A}$]*
If t is a $(0,t)$-tensor, define the alternation operator $\boldsymbol{A}$ acting on t by

$$\boldsymbol{A}(t)(v_1, ..., v_p) = \frac{1}{p!} \sum_{\rho \in S_p} sgn(\rho) t(v_{\rho(1)}, ..., v_{\rho(p)}),$$

where $sgn(\rho)$ is the sign of the permutation ρ,

$$sgn(\rho) = \begin{cases} +1, & \rho \text{ is even;} \\ -1, & \rho \text{ is odd.} \end{cases}$$

and S_p is the group of all permutations of the set $\{1, 2, ..., p\}$. The operator $\boldsymbol{A}$ is therefore skew-symmetrizes p-multilinear maps.

Definition 28. *[10][Wedge Product]*
If α is a k-form and β is a l-form on M, their wedge product $\alpha\wedge\beta$ is the $(k+l)$-form on M defined by

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!}\boldsymbol{A}(\alpha \otimes \beta).$$

For example, if α and β are one-forms, then

$$\alpha \wedge \beta(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1),$$

while if α is a 2-form and β is a 1-form, then

$$\alpha \wedge \beta(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) + \alpha(v_3, v_1)\beta(v_2) + \alpha(v_2, v_3)\beta(v_1).$$

Proposition 3.4.1. *[10] The wedge product has the following properties:*

(i) $\alpha \wedge \beta$ *is* ***associative****:* $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.

(ii) $\alpha \wedge \beta$ *is* ***homogeneous****:* $(a\alpha) \wedge \beta = a(\alpha \wedge \beta) = \alpha \wedge (a\beta)$.

(iii) $\alpha \wedge \beta$ *is* ***distributive*** *in* α, β*:*

$$(a\alpha_1 + b\alpha_2) \wedge \beta = a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta),$$

$$\alpha \wedge (a\beta_1 + b\beta_2) = a(\alpha \wedge \beta_1) + b(\alpha \wedge \beta_2).$$

(iv) $\alpha \wedge \beta$ *is* ***anticommutative****:* $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$*, where* α *is a* k*-form and* β *is an* l*-form.*

(v) Let α *be a* $k-$*form, if* k *is odd then* $\alpha \wedge \alpha = 0$.

(vi) In any chart,

$$(dx^{i_1} \wedge ... \wedge dx^{i_k}) \wedge (dx^{j_1} \wedge ... \wedge dx^{j_l}) = (dx^{i_1} \wedge ... \wedge dx^{i_k} \wedge dx^{j_1} \wedge ... \wedge dx^{j_l}).$$

Proof. The homogeneity and distributivity properties of the wedge product are immediate from Definition 28. From this we can deduce the following expression for the wedge product in local coordinates:

For $\alpha = \sum_{i_1,\ldots,i_k} \alpha_{i_1\ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ and $\beta = \sum_{j_1,\ldots,j_l} \beta_{j_1\ldots j_l} dx^{j_1} \wedge \ldots \wedge dx^{j_l}$

$$\alpha \wedge \beta = \frac{1}{k!l!} \sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} \alpha_{i_1\ldots i_k}\beta_{j_1\ldots j_l} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}.$$

The associativity property can now be checked straightforwardly. We derive the anticommutativity property (iv):

If $\alpha = \sum \alpha_{i_1\ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ and $\beta = \sum \beta_{j_1\ldots j_l} dx^{j_1} \wedge \ldots \wedge dx^{j_l}$, then

$$\begin{aligned}
\alpha \wedge \beta &= \frac{1}{k!l!} \sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} \alpha_{i_1\ldots i_k}\beta_{j_1\ldots j_l} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}. \\
&= \frac{(-1)^k}{k!l!} \sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} \alpha_{i_1\ldots i_k}\beta_{j_1\ldots j_l} dx^{j_1} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_2} \wedge \ldots \wedge dx^{j_l}. \\
&= \frac{(-1)^{kl}}{k!l!} \sum_{i_1,\ldots,i_k,j_1,\ldots,j_l} \alpha_{i_1\ldots i_k}\beta_{j_1\ldots j_l} dx^{j_1} \wedge \ldots \wedge dx^{j_l} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}. \\
&= (-1)^{kl} \beta \wedge \alpha.
\end{aligned}$$

For (v). $\alpha = \sum \alpha_{i_1\ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ where k odd, then

$$\alpha \wedge \alpha = (-1)^{k^2} \alpha \wedge \alpha = 0$$

Finally, we derive the(vi). Choose a chart about x. Then

$$(dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \wedge (dx^{j_1} \wedge \ldots \wedge dx^{j_l})$$

$$\begin{aligned}
&= \frac{(k+l)!}{k!l!} \mathcal{A}((dx^{i_1} \wedge \ldots \wedge dx^{i_k}) \otimes (dx^{j_1} \wedge \ldots \wedge dx^{j_l}). \\
&= \frac{(k+l)!}{k!l!} \mathcal{A}\Bigg(\sum_{\sigma \in S_k, \tau \in S_l} sgn(\sigma) sgn(\tau)\ dx^{i_{\sigma(1)}} \otimes \ldots \otimes dx^{i_{\sigma(k)}} \otimes dx^{j_{\tau(1)}} \otimes \ldots \otimes dx^{j_{\tau(l)}} \Bigg). \\
&= \frac{1}{k!l!} \sum_{\sigma \in S_k, \tau \in S_l} sgn(\sigma) sgn(\tau)\ dx^{i_{\sigma(1)}} \wedge \ldots \wedge dx^{i_{\sigma(k)}} \wedge dx^{j_{\tau(1)}} \wedge \ldots \wedge dx^{j_{\tau(l)}}. \\
&= dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}.
\end{aligned}$$

■

Theorem 3.1. *[5] Differential forms $\alpha_1, \alpha_2, ..., \alpha_r$ are linearly dependent if and only if their wedge product vanishes,*

$$\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_r = 0$$

Proof. If the differential forms are linearly dependent then without loss of generality we may assume that α_1 is a linear combination of the others,

$$\alpha_1 = a^2\alpha_2 + a^3\alpha_3 + ... + a^r\alpha_r.$$

Hence

$$\begin{aligned} \alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_r &= \left(\sum_{i=2}^{r} a^i\alpha_i\right) \wedge \alpha_2 \wedge ... \wedge \alpha_r \\ &= \sum_{i=2}^{r} \pm a^i \alpha_2 \wedge ... \wedge \alpha_i \wedge \alpha_i \wedge ... \wedge \alpha_r \\ &= 0 \end{aligned}$$

Conversely, suppose $\alpha_1, \alpha_2, ..., \alpha_r$ are linearly independent, then there exists a basis $\{e_j\}$ such that

$$e_1 = \alpha_1, \quad e_2 = \alpha_2, \quad .... , e_r = \alpha_r.$$

Since $e_1 \wedge e_2 \wedge ... \wedge e_r$ is a basis vector it cannot vanish. ∎

3.5 Examples of Algebraic Computation of Products

Example 3.2. *[5]Consider the $1-$forms $\alpha = xdx + ydy$, $\beta = ydx + xdy$.*

$$\begin{aligned} \alpha \wedge \beta &= (xdx + ydy) \wedge (ydx + xdy) \\ &= xydx \wedge dx + x^2dx \wedge dy + y^2dy \wedge dx + xydy \wedge dy. \\ &= (x^2 - y^2)dx \wedge dy. \end{aligned}$$

Example 3.3. *[5]Let* $\alpha = xdx + ydy$ *be a* $1-$*form,* $\beta = xzdx \wedge dz + yzdy \wedge dz$ *be a* $2-$*form.*

$$\begin{aligned} \alpha \wedge \beta &= (xdx + ydy) \wedge (xzdx \wedge dz + yzdy \wedge dz) \\ &= x^2 zdx \wedge dx \wedge dz + xyzdx \wedge dy \wedge dz \\ &+ yxzdy \wedge dx \wedge dz + y^2 zdy \wedge dy \wedge dz. \\ &= (xyz - xyz)dx \wedge dy \wedge dz = 0. \end{aligned}$$

Hence,

$$(xdx + ydy) \quad and \quad (xzdx \wedge dz + yzdy \wedge dz)$$

are linearly dependent.

3.6 Exterior Derivative

The exterior derivative $d\alpha$ of a k-form α on a manifold M is the $(k+1)$-form on M determined by the following proposition:

Proposition 3.6.1. *[10]There is a unique mapping d from k-forms on M to* $(k+1)$*-forms on M such that:*

(i) If α *is a* 0*-form* $(k = 0)$*, then df is the one-form that is the differential of f.*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

(ii) If α *and* β *are k-form fields, and* c_1 *and* c_2 *are constants, then*

$$d(c_1\alpha + c_2\beta) = c_1(d\alpha) + c_2(d\beta).$$

Taking an exterior derivative thus is a linear operation.

(iii) If α *is a k-form given in coordinates by*

$$\alpha = \sum \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \quad (sum\ on\ i_1 < \dots < i_k),$$

then the coordinate expression for the exterior derivative is

$$d\alpha = \sum \frac{\partial \alpha_{i_1 \ldots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \quad (\textit{sum on all } j \textit{ and } i_1 < \ldots < i_k) \tag{3.6.1}$$

(iv) The $(k+2)-$form $d^2 f = d(df)$ obtained by taking the exterior derivative of a $k-$form f twice is a constant form having the value 0 (a zero form).

$$\begin{aligned} d(df) &= \sum_i d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i = \sum_i \left(\sum_j \frac{\partial^2 f}{\partial x^i \partial x^j} dx^j\right) \wedge dx^i \\ &= \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i}\right) dx^i \wedge dx^j = 0. \end{aligned}$$

A k-form is called **closed** if $d\alpha = 0$ and **exact** if there is a $(k-1)$-form β such that $\alpha = d\beta$, we get the fundamental and remarkable property of exterior differentiation :

Proposition 3.6.2. *[10][Poincaré's Lemma]*
A closed form is locally exact; that is, if $d\alpha = 0$, there is a neighborhood about each point on which $\alpha = d\beta$.

Theorem 3.2. *[5][Cartan's Magic Formula]*
The exterior derivative of the wedge product of a k-form α and an l-form β is given by

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).$$

Proof.
By the definition of wedge product it suffices to show the rule for elementary form

$$\alpha = \sum_{i_1,\ldots,i_k} \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \quad \textit{and} \quad \beta = \sum_{j_1,\ldots,j_k} \beta_{j_1 \ldots j_l} dx^{j_1} \wedge \ldots \wedge dx^{j_l}.$$

and their wedge product

$$\alpha \wedge \beta = \sum_{\substack{i_1,\ldots,i_k \\ j_1,\ldots,j_l}} \alpha_{i_1 \ldots i_k} \beta_{j_1 \ldots j_l} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l}.$$

$$
\begin{aligned}
d(\alpha\wedge\beta) &= \sum\sum_{h=1}^{n}\frac{\partial(\alpha_{i_1\ldots i_k}\beta_{j_1\ldots j_l})}{\partial x^h}dx^h\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_l}.\\
&=\sum_{h=1}^{n}\left(\frac{\partial\alpha_{i_1\ldots i_k}}{\partial x^h}\beta_{j_1\ldots j_l}+\frac{\partial\beta_{j_1\ldots j_l}}{\partial x^h}\alpha_{i_1\ldots i_k}\right)dx^h\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_l}.\\
&=\left(\sum_{h=1}^{n}\frac{\partial\alpha_{i_1\ldots i_k}}{\partial x^h}dx^h\wedge dx^{i_1}\wedge\ldots\wedge dx^{i_k}\right)\wedge\left(\sum\beta_{j_1\ldots j_l}dx^{j_1}\wedge\ldots\wedge dx^{j_l}\right).\\
&+\left(\sum\alpha_{i_1\ldots i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k}\right)\wedge(-1)^k\left(\sum_{h=1}^{n}\frac{\partial\beta_{j_1\ldots j_l}}{\partial x^h}dx^h\wedge dx^{j_1}\wedge\ldots\wedge dx^{j_l}\right).\\
&=(d\alpha)\wedge\beta+(-1)^k\alpha\wedge(d\beta).
\end{aligned}
$$

■

3.7 Examples of Algebraic Computation of Derivatives

Example 3.4. *[5] If $\alpha=\sum_{i=1}^{n}f_i dx_i$ is a $1-$form on $\mathbb{R}^n$, then*

$$
\begin{aligned}
d\alpha &= \sum_{i=1}^{n}df_i\wedge dx_i=\sum_{i,j=1}^{n}\frac{\partial f_i}{\partial x_j}dx_j\wedge dx_i\\
&= \sum_{1\le i\le j\le n}\frac{\partial f_i}{\partial x_j}dx_j\wedge dx_i+\sum_{1\le j\le i\le n}\frac{\partial f_i}{\partial x_j}dx_j\wedge dx_i\\
&= -\sum_{1\le i\le j\le n}\frac{\partial f_i}{\partial x_j}dx_i\wedge dx_j+\sum_{1\le i\le j\le n}\frac{\partial f_j}{\partial x_i}dx_i\wedge dx_j \qquad (3.7.1)\\
&= \sum_{1\le i\le j\le n}\left(\frac{\partial f_j}{\partial x_i}-\frac{\partial f_i}{\partial x_j}\right)dx_i\wedge dx_j
\end{aligned}
$$

where in (3.7.1) in the first sum we used the alternating property and in the second sum we interchanged the roles of i and j.

Example 3.5. *[5] If $\alpha = \sum_{1\le i\le j\le n} f_{i,j} dx_i \wedge dx_j$ is a 2−form on $\mathbb{R}^n$, then*

$$\begin{aligned}
d\alpha &= \sum_{1\le i\le j\le n} df_{i,j} \wedge dx_i \wedge dx_j = \sum_{1\le i\le j\le n}\sum_{k=1}^{n} \frac{\partial f_{i,j}}{\partial x_k} dx_k \wedge dx_i \wedge dx_j \\
&= \sum_{1\le k<i<j\le n} \frac{\partial f_{i,j}}{\partial x_k} dx_k \wedge dx_i \wedge dx_j + \sum_{1\le i<k<j\le n} \frac{\partial f_{i,j}}{\partial x_k} dx_k \wedge dx_i \wedge dx_j \\
&+ \sum_{1\le i<j<k\le n} \frac{\partial f_{i,j}}{\partial x_k} dx_k \wedge dx_i \wedge dx_j \\
&= \sum_{1\le i<j<k\le n} \frac{\partial f_{j,k}}{\partial x_i} dx_i \wedge dx_j \wedge dx_k + \sum_{1\le i<j<k\le n} \frac{\partial f_{i,k}}{\partial x_j} dx_j \wedge dx_i \wedge dx_k \\
&+ \sum_{1\le i<j<k\le n} \frac{\partial f_{i,j}}{\partial x_k} dx_k \wedge dx_i \wedge dx_j . \qquad (3.7.2)
\end{aligned}$$

We remark that that the last equation can be simplified to

$$d\alpha = \sum_{1\le i<j<k\le n} \left(\frac{\partial f_{i,j}}{\partial x_k} - \frac{\partial f_{i,k}}{\partial x_j} + \frac{\partial f_{j,k}}{\partial x_i} \right) dx_i \wedge dx_j \wedge dx_k . \qquad (3.7.3)$$

Here in (3.7.2) we rearranged the subscripts(for instance, in the first term we relabelled $k \to i$, $i \to j$, $j \to k$) and in (3.7.3) we applied the alternating property.

Example 3.6. *[5]Let $\alpha = xydx - xydy + xy^2z^3dz$*

$$\begin{aligned}
d(xydx - xydy + xy^2z^3dz) =& d(xy) \wedge dx - d(xy) \wedge dy + d(xy^2z^3) \wedge dz. \\
=& (ydx + xdy) \wedge dx - (ydx + xdy) \wedge dy \\
+& (x(3y^2z^2dz + 2yz^3dy) + y^2z^3dx) \wedge dz. \\
=& (ydx + xdy) \wedge dx - (ydx + xdy) \wedge dy \\
+& (3xy^2z^2dz + 2xyz^3dy + y^2z^3dx) \wedge dz. \\
=& ydx \wedge dx + xdy \wedge dx - ydx \wedge dy - xdy \wedge dy \\
+& 3xy^2z^2dz \wedge dz + 2xyz^3dy \wedge dz + y^2z^3dx \wedge dz. \\
=& xdy \wedge dx - ydx \wedge dy + 2xyz^3dy \wedge dz + y^2z^3dx \wedge dz. \\
=& (-x - y)dx \wedge dy + y^2z^3dx \wedge dz + 2xyz^3dy \wedge dz.
\end{aligned}$$

Example 3.7. *[5] In this example we find the exterior derivative of the 2−form*

$$\alpha = x^2(y+z^2)dx \wedge dy + z(x^3+y)dy \wedge dz$$

$$\begin{aligned} d(x^2(y+z^2)dx \wedge dy + z(x^3+y)dy \wedge dz) =& d(x^2(y+z^2)) \wedge dx \wedge dy \\ &+ d(z(x^3+y)) \wedge dy \wedge dz. \\ =& 2zx^2 dz \wedge dx \wedge dy + 3zx^2 dx \wedge dy \wedge dz. \\ =& 5zx^2 dx \wedge dy \wedge dz. \end{aligned}$$

3.8 Lie Derivative

The Lie derivative can also be defined on differential forms. In this context, it is closely related to the exterior derivative. Both the Lie derivative and the exterior derivative attempt to capture the idea of a derivative in different ways. These differences can be bridged by introducing the idea of an anti-derivation or equivalently an interior product, after which the relationships fall out as a set of identities.

Definition 29. *[4][Interior Product]*
Let ω be an $k-$form and X be a vector field on M. Define the interior product

$$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$$

by

$$(\iota_X \omega)(Y_1, ..., Y_{k-1}) = \omega(X, Y_1, ..., Y_{k-1})$$

The differential form $\iota_X\omega$ is also called the contraction of ω with X.

That is, ι_X is $\mathbf{R}-$linear,and

$$\iota_X(\omega \wedge \eta) = (\iota_X\omega) \wedge \eta + (-1)^k \omega \wedge \iota_X \eta$$

The Lie derivative of an ordinary function f is just the contraction of the exterior derivative with the vector field X

$$\mathcal{L}_X f = \iota_X df$$

Lemma 3.3. *[4] Let ω be a differential form of degree l and X be any vector field, then the Lie derivative has the following properties:*

1. *$\mathcal{L}_X\omega$ is of the same degree as ω.*
2. *$d(\mathcal{L}_X\omega) = \mathcal{L}_X d\omega$.*
3. *$\mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$.*
4. *$\mathcal{L}_x(\omega \wedge \theta) = \mathcal{L}_X\omega \wedge \theta + \omega \wedge \mathcal{L}_X\theta$.*
5. *$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega$*

where ι is the interior product between ω and X and d is the exterior derivative.

Theorem 3.4. *[4] The differential $1-$form*

$$\omega(x) = \sum_{i=1}^{n} \omega_i(x)dx^i$$

is $k-$homogeneous if and only if

$$\mathcal{L}_X\omega = (k+1)\omega$$

where $X = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$.

Proof. Let ω and X be defined as above. Then, recall that

$$\mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$$

We calculate each term on the right hand side. It follows that

$$\iota_X d\omega = \sum_{i,j=1}^{n} \frac{\partial\omega_i}{\partial x^j} x^j dx^i - \sum_{i,j=1}^{n} \frac{\partial\omega_i}{\partial x^j} x^i dx^j. \tag{3.8.1}$$

Similarly, we have

$$\iota_X\omega = \sum_i \omega_i(x)x^i$$

Then

$$d(\iota_X\omega) = \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} x^i dx^j + \sum_i \omega_i(x) dx^i \tag{3.8.2}$$

From (3.8.1) and (3.8.2), we get

$$\mathcal{L}_X\omega = \sum_{i,j=1}^{n} \frac{\partial \omega_i}{\partial x^j} x^j dx^i + \sum_i \omega_i(x) dx^i$$

By euler's equation, we conclude that $\omega(x)$ is $k-$homogeneous if and only if

$$\mathcal{L}_X\omega = (k+1)\omega.$$

■

3.9 Integrability of Homogeneous Differential Forms

Lemma 3.5. *[4] If ω is a C^1, $k-$homogeneous $1-$form such that $\iota_X\omega = 0$ then*

$$\iota_X d\omega = (k+1)\omega.$$

Proof. $k-$homogeneity implies that

$$\mathcal{L}_X d\omega = (k+1)\omega.$$

On the other hand

$$(k+1)\omega = \mathcal{L}_X\omega = \iota_X d\omega + d(\iota_X\omega)$$

since $\iota_X\omega = 0$,

$$\iota_X d\omega = (k+1)\omega.$$

■

Proposition 3.9.1. *[4] Let ω be a C^1 differential $1-$form such that $\iota_X\omega = 0$. Then $\omega \wedge d\omega = 0$ with ω $k-$homogeneous if and only if there is a $1-$form β such that $d\omega = \beta \wedge \omega$ with $\iota_X\beta = k+1$*

Proof. If $d\omega = \beta \wedge \omega$, then

$$\omega \wedge d\omega = 0.$$

Moreover,

$$\iota_X d\omega = (\iota_X \beta)\omega - \beta(\iota_X \omega) = (k+1)\omega,$$

but

$$\mathcal{L}_X d\omega = \iota_X d\omega + d(\iota_X \omega) = (k+1)\omega.$$

This proves the $k-$ homogeneity of ω. Conversely, if $\omega \wedge d\omega = 0$, then there exists a $1-$form β such that

$$d\omega = \beta \wedge \omega.$$

Hence,

$$\iota_X d\omega = (\iota_X \beta)\omega - \beta(\iota_X \omega)$$

by using Lemma 3.5,

$$\iota_X d\omega = (\iota_X \beta)\omega = (k+1)\omega.$$

Then

$$\iota_X \beta = k+1.$$

■

Theorem 3.6. *[4] Let ω be a C^1, $k-$homogeneous differential $1-$ form such that $\omega \wedge d\omega = 0$ in a neighbourhood $\mathcal{U}$ of some point $\bar{x}$. Then, there exists a $(k+1)-$homogeneous function f and a $0-$homogeneous function g, defined in a possibly smaller neighbourhood $\mathcal{V} \subset \mathcal{U}$ such that $\omega(x) = f(x)dg(x)$.*

Proof. Suppose that $\omega \wedge d\omega = 0$. Then, there exist two functions f and g such that $\omega = f dg$.
Since

$$\iota_X \omega = 0,$$

then

$$\iota_X dg = 0;$$

that is, g is $0-$homogeneous. We have also,

$$d\omega = df \wedge dg$$

and

$$dg = \omega / f.$$

It follows that

$$d\omega = \frac{df}{f} \wedge \omega.$$

Apply the vector field X to both sides of the previous equation and use Lemma (3.5) to get

$$(k+1)\omega = \iota_X \frac{df}{f} \omega.$$

Thus, $\iota_X df = (k+1)f$, which proves that $f(x)$ is $(k+1)-$homogeneous. ■

4. SINGLE CONSTRAINT AND NON-HOMOGENEOUS MODELS

The standard utility maximization problem under the budget constraint takes the form

$$\mathcal{P}\left\{\max_x U(x) \quad ; \quad p^T x = w(p),\right.$$

where U is utility function that satisfies certain smoothness, monotonicity, and concavity conditions, and $w(p)$ is convex and homogeneous of degree one. The solution of the above maximization problem was characterized in Aloqeili[2]. We state here the result that concerns the homogeneous case.

Theorem 4.1. *Let $x(p)$ be given, and define $p^T x(p) = w(p)$. Suppose $w(p)$ is convex and homogeneous of degree one. Then, there exist function $\lambda(p)$ and $U(x)$ such that $D_x U(x(p)) = \lambda(p)p$ in the neighbourhood of a point $\bar{p}$ if and only if there is some vector $\beta(p)$ with*

$$p^T \beta(p) = 1$$

such that for all i, j we have:

$$\frac{\partial x^i}{\partial p_j} - \beta^i \sum_k \frac{\partial x^k}{\partial p_j} p_k = \frac{\partial x^j}{\partial p_i} - \beta^j \sum_k \frac{\partial x^k}{\partial p_i} p_k$$

in a neighbourhood of $\bar{p}$

In the general case, we need to characterize the solution of a multi-constraint optimization problem that takes the form:

$$\mathcal{P}\left\{\max_x f(x) \quad ; \quad Ax \leq C(A),\right.$$

where f is a function satisfies the following conditions:

1. f is strictly increasing with respect to each coordinate of the vector x.
2. the Hessian matrix $D_x^2 f$ is negative definite on the subspace $\{D_x f\}^{\perp}$.
3. f is of class C^2.

and for each $i \in \{1, ..., m\}$, the function C^i has the following properties

1. C^i :$\mathbf{R}^{\mathbf{n}}_{++} \to \mathbf{R}_{++}$.
2. C^i is a convex function of a^i.
3. C^i is of class C^2.
4. C^i is not homogeneous of degree one in a^i; that is, $(a^i)^T D_{a^i} C^i - C^i(a^i) \neq 0$.

Aloqeili[3], derived necessary and sufficient conditions for a given function to be the solution of this problem.

Define a family of 1-forms Ω_k, $k = 1, ..., m$, by

$$\Omega_k = \sum_{s=1}^{m} \lambda_{sk} \omega^s,$$

where ω^s is the 1-form defined by

$$\omega^i = \sum_{j=1}^{n} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) da^i_j$$

and λ_{sk} are given functions, x is a solution of the above multi-constraint problem.

Theorem 4.2. *Given the family of 1-form $\Omega_1, ..., \Omega_m$. Then $\Omega_k \wedge d\Omega_k = 0$ if and only if for any $k' \in \{1, ..., m\}$, the following conditions are satisfied for all $1 \leq i, s \leq m$, $1 \leq j, l \leq n$.*

$$\frac{\partial \lambda_{ik}}{\partial a^s_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^s_l} + \frac{\eta_{k'}}{\lambda_{kk'}} \left(\sum_{j'} \frac{\partial \lambda_{sk}}{\partial a^{k'}_{j'}} a^{k'}_{j'} \sum_r \frac{\partial x^r}{\partial a^s_l} a^s_r - \lambda_{sk} \sum_{j'} \frac{\partial x^l}{\partial a^{k'}_{j'}} a^{k'}_{j'} \right.$$

$$-\frac{1}{\eta_{k'}}\frac{\partial\lambda_{k'k}}{\partial a_l^s}+\lambda_{k'k}\sum_j\frac{\partial x^{j'}}{\partial a_l^s}a_{j'}^{k'}\Big)\lambda_{ik}\sum_{r=1}^n\frac{\partial x^r}{\partial a_j^i}a_r^i$$

$$=\frac{\partial\lambda_{sk}}{\partial a_j^i}\sum_r\frac{\partial x^r}{\partial a_l^s}a_r^s-\lambda_{sk}\frac{\partial x^l}{\partial a_j^i}+\frac{\eta_{k'}}{\lambda_{kk'}}\Big(\sum_{j'}\frac{\partial\lambda_{ik}}{\partial a_{j'}^{k'}}a_{j'}^{k'}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i-\lambda_{ik}\sum_{j'}\frac{\partial x^j}{\partial a_{j'}^{k'}}a_{j'}^{k'}$$

$$-\frac{1}{\eta_{k'}}\frac{\partial\lambda_{k'k}}{\partial a_j^i}+\lambda_{k'k}\sum_{j'}\frac{\partial x^{j'}}{\partial a_j^i}a_{j'}^{k'}\Big)\lambda_{sk}\sum_{r=1}^n\frac{\partial x^r}{\partial a_l^s}a_r^s \tag{4.0.1}$$

Let $x(A)$ be a solution of problem, $C(A)=Ax(A)$, then $n\times n$ matrix M^i defined by

$$M_{jl}^i=\sum_r\frac{\partial^2x^r}{\partial a_l^i\partial a_j^i}a_r^i+\frac{\partial x^l}{\partial a_j^i}+\frac{\partial x^j}{\partial a_l^i}$$

and T^i be the matrix defined by

$$T_{jl}^i=\sum_r\frac{\partial^2x^r}{\partial a_l^i\partial a_j^i}a_r^i+\frac{\partial x^l}{\partial a_j^i}\ .$$

Theorem 4.3. *Let $x(A)\in\mathbf{R}_{++}^n$, $\lambda_{ik}(A)>0$ be given functions defined on a neighbourhood $\mathcal{U}$ of some point $\bar{A}\in\mathbf{R}_{++}^{mn}$. Define $C(A)=Ax(A)$. Suppose that the following conditions are satisfied in $\mathcal{U}$ for all $i,k=1,...,m$:*

1. *$\lambda_{ti}\lambda_{sk}=\lambda_{si}\lambda_{tk}$ for all $1\le i,k,s,t\le m$.*
2. *Conditions (4.0.1)*
3. *The matrix M^i is positive semi-definite.*
4. *The restriction of the matrix T^i to $\{(a^i)^TD_{a^i}x\}^\perp$ is a positive definite.*

Then, there exist positive functions $\lambda_1,.....,\lambda_m$ and a function V which is quasi-convex with respect to a^i for each i, defined in a neighbourhood $\mathcal{V}\subset\mathcal{U}$ such that $D_{a^i}V=\lambda_i(D_{a^i}C^i-x)$.

5. SOLUTION OF THE INVERSE PROBLEM-MAIN RESULTS

5.1 Setting up The Model

We consider a multi-constraint maximization problem of the form

$$(\mathcal{P})\left\{\ \max_x f(x) \quad ; \quad Ax = C(A),\right.$$

where f is a function that satisfies certain regularity and convexity conditions that are specified later, A is an $m \times n$ matrix of rank m and $C : \mathbb{R}^{m\times n}_{++} \to \mathbb{R}^m_{++}$ is a given mapping. The i^{th} constraint takes the form $(a^i)^T x = C^i(A)$ where a^i is the i^{th} row of the matrix A. Define the Lagrangian function

$$L(x,\lambda) = f(x) + \sum_{k=1}^{m} \lambda_k \left(C^k(A) - \sum_{l=1}^{n} a^k_l x^l \right)$$

with $x \in \mathbb{R}^n_{++}$, and $\lambda \in \mathbb{R}^m_{++}$. The first order conditions for interior maximum give

$$\frac{\partial f}{\partial x^j} = \sum_{k=1}^{m} \lambda_k a^k_j, \qquad j = 1, ..., n$$

$$Ax = C(A).$$

Define the value function of this problem by

$$V(A) = \max_x \left\{\ f(x) + \sum_{k=1}^{m} \lambda_k \left(C^k(A) - \sum_{l=1}^{n} a^k_l x^l \right) \right\}. \tag{5.1.1}$$

If the functions $C^1(a^1), ..., C^m(a^m)$ are convex on $\mathbb{R}^n_{++}$ then the value function $V(a^1, ..., a^m)$ is quasi-convex with respect to each a^i for $i = 1, ..., m$.

Differentiating the function $V(A)$ with respect to a^i_j and using the envelope theorem we get

$$\frac{\partial V}{\partial a^i_j} = \sum_{k=1}^{m} \lambda_k \frac{\partial C^k}{\partial a^i_j} - \lambda_i x^j. \tag{5.1.2}$$

We suppose that C^k is a function of the vector $a^k \in \mathbb{R}^n_{++}$ only, where a^k is the k^{th} row of the matrix A. Moreover, we assume that each component of the mapping $C(A)$ is homogeneous of degree one. This implies, in particular that, the functions $x(A)$ and $V(A)$ are homogeneous of degree zero and the Lagrange multiplier corresponding to the i^{th} constraint, $\lambda_i(A)$ is homogeneous of degree -1 in a^i, and of degree 0 in a^k for $i \neq k$.
We adopt the following assumptions on the mapping C.

Assumption 5.1. *For each $i \in \{1, ..., m\}$, we assume that the function C^i has the following properties:*

1. $C^i : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ *is a function of a^i only.*
2. C^i *is a convex function of a^i.*
3. C^i *is of class C^2.*
4. C^i *is homogeneous of degree one in a^i; that is, $(a^i)^T D_{a^i} C^i - C^i(a^i) = 0$.*

We consider the following assumptions on the objective function f.

Assumption 5.2. *Assume the function f satisfies the following conditions:*

1. f *is strictly increasing with respect to each coordinate of the vector x.*
2. *the Hessian matrix $D^2_x f$ is negative definite on the subspace $\{D_x f\}^{\perp}$.*
3. f *is of class C^2.*

By applying the implicit function theorem, we show that the solution of the above maximization problem as well as the associated vector of Lagrange multipliers are of class C^2.

Assumption (1) implies that $D_{a^i}C^k = 0$ if $i \neq k$ which reduces equation (5.1.2) to

Lemma 5.1. *The partial derivative of the value function $V(A)$ with respect to a^i_j is given by*

$$\frac{\partial V}{\partial a^i_j} = \lambda_i \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right). \tag{5.1.3}$$

Define a family of differential 1-forms $\omega^1, ..., \omega^m$ by

$$\omega^i = \sum_{j=1}^{n} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) da^i_j. \tag{5.1.4}$$

The differential of V is given by

$$dV = \sum \frac{\partial V}{\partial a^i_j} da^i_j.$$

It follows that the differential of V, can be written as:

$$dV = \sum_{i=1}^{m} \lambda_i \omega^i.$$

Notice that

$$d\omega^i = \sum_{j,l} \frac{\partial^2 C^i}{\partial a^i_l \partial a^i_j} da^i_l \wedge da^i_j - \sum_{j,k,l} \frac{\partial x^j}{\partial a^k_l} da^k_l \wedge da^i_j.$$

The coefficients in the first summation are symmetric, so we end up with

$$d\omega^i = -\sum_{j,k,l} \frac{\partial x^j}{\partial a^k_l} da^k_l \wedge da^i_j.$$

The i^{th} constraint is $(a^i)^T x(A) = C^i(a^i)$. Differentiating both sides of this equality with respect to a^i_j

$$\frac{\partial C^i}{\partial a^i_j} = \sum_{r=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_r + x^j$$

and rearranging the above formula, we get:

$$\frac{\partial C^i}{\partial a^i_j} - x^j = \sum_{r=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_r. \tag{5.1.5}$$

Using this result, the 1-form ω^i can be written as

$$\omega^i = \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_r da^i_j \tag{5.1.6}$$

Now, our inverse problem can be stated as follows:

- We observe the function $x^j(A), j = 1, ..., n$ from $\mathbb{R}^{mn}_{++}$ to $\mathbb{R}_{++}$.
- Then we define the functions $C^i(a^i) = (a^i)^T x(A)$.
- We observe also a family of positive functions λ_{ik} using symmetry conditions that will be given below.
- Our goal is to find a function $f(x)$, by first finding the value function $V(A)$, such that $x(A) \in \text{argmax}\ \{f(x)|Ax = C(A)\}$ and $V(A) = f(x(A))$.

To allow for better follow up of our exposition, we will restrict the ranges of the subscripts and superscripts used in the sequel as follows, $1 \leq i, k, k', s, t \leq m$ and $1 \leq j, j', l, l', r \leq n$. In what follows, δ^i_k denotes the Kronecker symbol which equals one if $i = k$ and zero otherwise.

5.2 Preliminary Results

In this section, we give some important preliminary results that will be used to solve the inverse problem.

Theorem 5.2. *The family of functions λ_i, $i, k = 1, ..., m$ have the following homogeneity properties:*

- *Homogeneous of degree -1 in* a^i.
- *Homogeneous of degree 0 in* a^k $, k \neq i$.

Proof.

$$\frac{\partial V}{\partial a^i_j} = \lambda_i \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right).$$

The function $\frac{\partial V}{\partial a^i_j}$ is homogeneous of degree -1 in a^i, and the functions $\frac{\partial C^i}{\partial a^i_j}$, x^j homogeneous of degree 0 in a^i. Thus λ_i is homogeneous of degree -1 in a^i and homogeneous of degree 0 in a^k, $k \neq i$. ■

Theorem 5.3. *Let* $V(A)$ *be the value function given in (5.1.1). Then,* $V(A)$ *has the following properties:*

(a) Positively homogeneous of degree zero if $C(A)$ *is positively homogeneous of degree one.*

(b) Quasi-convex if $C(A)$ *is convex.*

Proof. (a) Suppose $C(A)$ is homogeneous of degree one, then $Ax = C(A)$ is equivalent to

$$tAx = tC(A) = C(tA).$$

We need to show $V(A) = V(tA)$, $\forall t > 0$. Note that

$$V(A) = \{\max \ f(x) \ \ s.t \ \ Ax = C(A)\}$$

$$V(tA) = \{\max \ f(x) \ \ s.t \ \ tAx = C(tA)\},$$

but

$$C(tA) = tC(A)$$

by homogeneity of degree one.

$$\begin{aligned} V(tA) &= \{\max \ f(x) \ \ s.t \ \ tAx = C(tA) = tC(A)\}. \\ &= \{\max \ f(x) \ \ s.t \ \ Ax = C(A)\}. \\ &= V(A). \end{aligned}$$

(b) We prove that the value function is quasi-convex. Let
$\hat{A} = (\hat{a}^1, ..., \hat{a}^m)^T$ and $\bar{A} = (\bar{a}^1, ..., \bar{a}^m)^T$. Consider the convex combinations:

$$\tilde{A} = (\tilde{a}^1, ..., \tilde{a}^m)^T = \kappa(\hat{a}^1, ..., \hat{a}^m)^T + (1-\kappa)(\bar{a}^1, ..., \bar{a}^m)^T$$

for $\kappa \in (0,1)$. Suppose that $V(\hat{A}) \leq U$ and $V(\bar{A}) \leq U$. We want to show that

$$V(\tilde{A}) \leq \max\{V(\hat{A}), V(\bar{A})\}.$$

Introduce the following sets

$$\hat{S} = \{x | \hat{A}x \leq C(\hat{A})\}, \quad \bar{S} = \{x | \bar{A}x \leq C(\bar{A})\}, \quad \tilde{S} = \{x | \tilde{A}x \leq C(\tilde{A})\}$$

We claim that $\tilde{S} \subset \hat{S} \cup \bar{S}$.

Indeed, if this is not the case, then there exists x such that $\hat{A}x > C(\hat{A})$ and $\bar{A}x > C(\bar{A})$ whereas $\tilde{A}x \leq C(\tilde{A})$. It follows that for any $\kappa \in (0,1)$, $\kappa\hat{A}x > \kappa C(\hat{A})$ and $(1-\kappa)\bar{A}x > (1-\kappa)C(\bar{A})$. Adding up the last two inequalities and using the convexity of $C(A)$, we get

$$\tilde{A}x = \kappa\hat{A} + (1-\kappa)\bar{A})x > \kappa C(\hat{A}) + (1-\kappa)C(\bar{A}) \geq C(\kappa\hat{A} + (1-\kappa)\bar{A}) = C(\tilde{A}).$$

Hence, $\tilde{A}x > C(\tilde{A})$ which is a contradiction, so $\tilde{S} \subset \hat{S} \cup \bar{S}$ as announced. This result implies that

$$V(\tilde{A}) = \max_{x \in \tilde{S}} f(x) \leq \max_{x \in \hat{S} \cup \bar{S}} f(x) = \max\{V(\hat{A}), V(\bar{A})\},$$

which means that $V(A)$ is quasi-convex.

■

Theorem 5.4. *If the function $f(x)$ satisfies Assumption (5.2) and $C(A)$ is of class C^2, then the map $A \to x(A)$ and the function $A \to \lambda(A)$ are of class C^2.*

Proof. Recall that

$$V(A) = \max_x \left\{ f(x) + \sum_{k=1}^{m} \lambda_k \left(C^k(A) - \sum_{l=1}^{n} a_l^k x^l \right) \right\}.$$

Derive with respect to x and λ,we get the first order conditions:

$$\begin{aligned} D_x f - A^T \lambda &= \overrightarrow{0}, \\ Ax - C(A) &= \overrightarrow{0}. \end{aligned} \tag{5.2.1}$$

Let F be defined by $F(A, x, \lambda) = (F_1(A, x, \lambda), F_2(a, x, \lambda))$, where $F_1(A, x, \lambda) = D_x f - A^T \lambda$, and $F_2(A, x, \lambda) = Ax - C(A)$.

$$D_{x,\lambda} F = \begin{pmatrix} D_x F_1 & D_\lambda F_1 \\ D_x F_2 & D_\lambda f_2 \end{pmatrix} = \begin{pmatrix} D_x^2 f(x) & -A^T \\ A & 0 \end{pmatrix}.$$

We need to show $D_{x,\lambda} F$ is nonsingular to apply implicit function theorem. Let $\zeta = (\zeta^1, \zeta^2)^T$ where $\zeta \in \mathbb{R}^{n+m}$. We will show that the linear system $(D_{x,\lambda} F)\zeta = \overrightarrow{0}$ has only the zero solution. Let $(\zeta^1, \zeta^2)^T \neq \overrightarrow{0}$,

$$\begin{pmatrix} D_x^2 f(x) & -A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$D_x^2 f(x) \zeta^1 - A^T \zeta^2 = \overrightarrow{0} \qquad (1)$$

$$A\zeta^1 = \overrightarrow{0} \qquad (2)$$

It follows that $\zeta^1 \in N(A) = \{\nabla f\}^\perp$ which we get be multiplying (5.2.1) from left by ${\zeta^1}^T$. Multiply the first equation by ${\zeta^1}^T$, we get

$${\zeta^1}^T (D_x^2 f(x)) \zeta^1 = 0$$

which is a contradiction to the assumption of the Hessian matrix of f is negative definite on $\{\nabla f\}^\perp$. Thus, $\zeta^1 = \overrightarrow{0}$ and $\zeta^2 = \overrightarrow{0}$, then the homogeneous system $D_{x,\lambda}F\zeta = 0$ has only the trivial solution $\zeta = (\zeta^1, \zeta^2)^T = (0,0)^T$, so the matrix $D_{x,\lambda}F$ is nonsingular and we can apply the implicit function theorem which guarantees that $x(A)$ and $\lambda(A)$ are of class C^2. ∎

Lemma 5.5. *Let $x(A)$ be a solution of a multi-constraint maximization problem of the above type, then*

1. $\sum_{l=1}^{n} \frac{\partial x^l}{\partial a^i_j} a^k_l = 0$ *if* $i \neq k$.
2. $\sum_{j=1}^{n} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) a^i_j = (a^i)^T (D_{a^i} x) a^i = 0.$

Proof. Differentiate the k^{th} constraint $(a^k)^T x = C^k(a^k)$ with respect to a^i_j, we get

$$\sum_{j=1}^{n} \frac{\partial x^l}{\partial a^i_j} a^k_l + x^j \delta^i_k = \frac{\partial C^k}{\partial a^i_j} \delta^i_k.$$

Condition (a) follows when $i \neq k$. If $i = k$, then multiply both sides of the last equality by a^i_j, summing over j, we find

$$\sum_{j} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) a^i_j = \sum_{j,r=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_j a^i_r.$$

Now use homogeneity of x to get (b). ∎

5.3 Mathematical Integration: Necessary and Sufficient Conditions

Our objective now is to give sufficient conditions and to express them as a system of partial differential equations that have to be satisfied by the coefficient function λ_{ik} and the function $x(A)$.

Definition 30 (Coefficient Function λ_{ik})**.**
Let λ_{ik} be the function defined by $\lambda_{ik} = \lambda_i/\lambda_k$ such that $\lambda_{ii} = 1$ for every $i = 1, ..., m$, $\lambda_{ik}\lambda_{ki} = 1$.

The following result follows from homogeneity properties of $\lambda_i,\ i = 1, ..., m$.

Lemma 5.6. *The family of functions $\lambda_{ik},\ i, k = 1, ..., m$ have the following homogeneity properties:*

a. *Homogeneous of degree -1 in a^i.*

b. *Homogeneous of degree 1 in a^k.*

c. *Homogeneous of degree 0 in $a^{k'}$, $k' \neq i, k$.*

That is,

$$\sum_l \frac{\partial \lambda_{ik}}{\partial a_l^{k'}} a_l^{k'} \quad = \quad \lambda_{ik}(\delta_k^{k'} - \delta_i^{k'}).$$

Equation (5.1.3) implies that

$$\frac{1}{\lambda_k}\frac{\partial V}{\partial a_j^i} = \lambda_{ik}\left(\frac{\partial C^i}{\partial a_j^i} - x^j\right). \tag{5.3.1}$$

Define a family of 1-forms Ω_k, $k = 1, ..., m$, by

$$\Omega_k = \sum_{s=1}^{m} \lambda_{sk}\omega^s, \tag{5.3.2}$$

where ω^s is the 1-form defined by (5.1.4) or the equivalent form (5.1.6). Notice that $\Omega_1, ..., \Omega_m$ are defined using observable functions only. Then, equation (5.3.1) can be written as $\mu_k dV = \Omega_k$ which is equivalent to $\Omega_k \wedge d\Omega_k = 0$. Clearly, the family of 1-forms defined by (5.3.2) are collinear to the same gradient dV. The last equation gives the necessary and sufficient conditions for mathematical integration. This result stems from the underlying structure of the optimization problem. The following result proves that the 1-forms $\Omega_1, ..., \Omega_m$ are proportional.

Lemma 5.7. *[3] Let $\Omega_1, ..., \Omega_m$ be the family of 1-forms defined by (5.3.2) with $\lambda_{ik} = \frac{\lambda_i}{\lambda_k}$, then*

$$\Omega_i \wedge \Omega_k = 0$$

for all $i, k = 1, ...m$.

Proof. Using the definition of Ω_k in (5.3.2) we have

$$\begin{aligned}\Omega_i \wedge \Omega_k &= \sum_{s,t=1}^{m} (\lambda_{ti}\lambda_{sk})\omega^t \wedge \omega^s. \\ &= \sum_{t<s} (\lambda_{ti}\lambda_{sk} - \lambda_{si}\lambda_{tk})\omega^t \wedge \omega^s.\end{aligned}$$

The coefficients $\lambda_{ti}\lambda_{sk} - \lambda_{si}\lambda_{tk}$ are identically zero since

$$\frac{\lambda_{ti}\lambda_{sk}}{\lambda_{si}\lambda_{tk}} = \frac{\lambda_t\lambda_s}{\lambda_i\lambda_k}\frac{\lambda_i\lambda_k}{\lambda_s\lambda_t} = 1.$$

∎

This is a general result that is true for any 1-forms defined by equation (5.3.2) with coefficient $\lambda_{ik} = \lambda_i/\lambda_k$. This result is obvious if $\Omega_k = \mu_k dV$.

Theorem 5.8. *[3] Given the family of 1-forms $\Omega_1, ..., \Omega_m$ defined above, then there exist m+1 functions $\mu_1, ..., \mu_m$ and V, defined in a neighbourhood $\mathcal{U}$ of some element $\bar{A} \in \mathbb{R}^{mn}_{++}$, such that $\Omega_k = \mu_k dV$ for $k = 1, ..., m$ if and only if the condition $\Omega_k \wedge d\Omega_k = 0$ holds in a neighbourhood $\mathcal{V}$ of $\bar{A}$ with $\mathcal{U} \subset \mathcal{V}$.*

Proof. Using Darboux Theorem [6], $\Omega_k \wedge d\Omega_k = 0$ if and only if there exist two functions μ_k and V_k such that

$$\Omega_k = \mu_k dV_k.$$

Lemma 5.7 implies that

$$\Omega_i \wedge \Omega_k = \mu_i\mu_k dV_i \wedge dV_k = 0.$$

Therefore,

$$dV_k = \phi_{ik}(A)dV_i \ , \quad \forall \ i, k = 1, ..., m$$

for some function ϕ_{ik}. So we can set

$$dV_1 = ... = dV_m = dV.$$

■

We also need the following lemma.

Lemma 5.9. *[3] Let $\Omega_1, ..., \Omega_m$ be the family of differential 1-forms defined in (5.3.2). Then, if $\Omega_i \wedge d\Omega_i = 0$ for some i, then*

$$\Omega_k \wedge d\Omega_k = 0$$

for any $k \in \{1, ..., m\}$.

Proof. Let $i, k \in \{1, ..., m\}$. Assume that

$$\Omega_i \wedge d\Omega_i = 0.$$

Note that $\Omega_i \wedge \Omega_k = 0$ if and only if $\Omega_k = \varphi\Omega_i$ for some function φ.
Taking the exterior derivative we get

$$d\Omega_k = \varphi d\Omega_i + d\varphi \wedge \Omega_i.$$

Multiply both sides of the last equation by Ω_k and using the fact that $\Omega_k = \varphi\Omega_i$, we find that

$$\Omega_k \wedge d\Omega_k = \varphi^2\Omega_i \wedge d\Omega_i + \varphi\Omega_i \wedge d\varphi \wedge \Omega_i = 0.$$

■

Clearly, the 1-forms $\Omega_1, ..., \Omega_m$ belong to the space of 1-forms spanned by $\omega^1, ..., \omega^m$. Moreover, it follows from the definition of $\omega^1, ..., \omega^m$ that they are linearly independent since $\omega^1 \wedge, ..., \wedge\omega^m \neq 0$. Let us consider the following result.

Lemma 5.10. *[3] Let $\beta_1, ..., \beta_m$ belong to the subspace of 1-forms spanned by $\alpha^1, ..., \alpha^m$. Suppose that $\alpha^1, ..., \alpha^m$ are linearly independent; that is, $\alpha^1 \wedge ... \wedge \alpha^m \neq 0$.*
Then $\beta_i \wedge \beta_k = 0$ if and only if there exist C_2^m rank-one symmetric $m \times m$ matrices $M_{ik} = (b_{is}b_{kt})$, such that $\beta_i = \sum_{s=1}^m b_{is}\alpha^s$.

Proof. Since $\beta_1, ..., \beta_m$ belong to the linear span of $\alpha^1, ..., \alpha^m$ then for any i there exist m functions $b_{i1}, ..., b_{im}$ such that

$$\beta_i = \sum_{s=1}^{m} b_{is}\alpha^s.$$

Therefore,

$$\beta_i \wedge \beta_k = \sum_{s,t} b_{is}b_{kt}\alpha^s \wedge \alpha^t = \sum_{s<t}(b_{is}b_{kt} - b_{it}b_{ks})\alpha^s \wedge \alpha^t.$$

Thus

$$\beta_i \wedge \beta_k = 0$$

if and only if

$$b_{is}b_{kt} = b_{it}b_{ks}.$$

■

Theorem 5.11. *Let $\Omega_1, ..., \Omega_m$ be the family of 1-forms, then for any $k = 1, ..., m$*

$$\Omega_k \wedge d\Omega_k = 0$$

if and only if there exists a 1-form α_k such that

$$d\Omega_k = \alpha_k \wedge \Omega_k. \tag{5.3.3}$$

Our objective now is to explicit the necessary and sufficient conditions for mathematical integration given in Theorem (5.8).

Theorem 5.12. *Given the family of 1-forms $\Omega_1, ..., \Omega_m$. Then*

$$\Omega_k \wedge d\Omega_k = 0$$

if and only if

1. *there exist a set of rank one $n \times m$ matrices $R_1, R_2, ..., R_m$ that satisfy the conditions $R_k(a^i)^T = \delta^i_k$.*

2. *for all* $1 \leq i, s \leq m$, $1 \leq j, l \leq n$ *the following conditions are satisfied.*

$$\frac{\partial \lambda_{ik}}{\partial a_l^s} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i - \lambda_{ik} \frac{\partial x^j}{\partial a_l^s} - R_{ks}^l(A) \lambda_{ik} \sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i$$
$$= \frac{\partial \lambda_{sk}}{\partial a_j^i} \sum_r \frac{\partial x^r}{\partial a_l^s} a_r^s - \lambda_{sk} \frac{\partial x^l}{\partial a_j^i} - R_{ki}^j(A) \lambda_{sk} \sum_r \frac{\partial x^r}{\partial a_l^s} a_r^s. \tag{5.3.4}$$

Proof. 1. Recall that $\Omega_k \wedge d\Omega_k = 0$ if and only if there exists a 1-form α_k such that

$$d\Omega_k = \alpha_k \wedge \Omega_k. \tag{5.3.5}$$

The 1-form α_k can be identified (mod Ω_k). Notice that

$$\Omega_k = \sum_{i=1}^m \lambda_{ik} \sum_{r,j=1}^n \frac{\partial x^r}{\partial a_j^i} a_r^i da_j^i.$$

Define a family of vector fields by

$$\xi^i = \sum_{j=1}^n a_j^i \frac{\partial}{\partial a_j^i}.$$

Then,

$$\begin{aligned} < \Omega_k, \xi^{k'} > &= < \sum_{i=1}^m \lambda_{ik} \sum_{r,j=1}^n \frac{\partial x^r}{\partial a_j^i} a_r^i da_j^i, \sum_{j=1}^n a_j^{k'} \frac{\partial}{\partial a_j^{k'}} > \\ &= \lambda_{k'k} \sum_{r,j=1}^n \frac{\partial x^r}{\partial a_j^{k'}} a_r^{k'} a_j^{k'} = 0. \end{aligned}$$

Using (5.1.5), we can write

$$< \Omega_k, \xi^{k'} >= \lambda_{k'k} \sum_{j=1}^n \left(\frac{\partial C^{k'}}{\partial a_j^{k'}} - x^j \right) a_j^{k'} = 0.$$

To find a 1-form α_k that satisfies the equation $d\Omega_k = \alpha_k \wedge \Omega_k$, we apply both sides of this equation to the vector field $\xi^{k'}$, so we have

$$< d\Omega_k, (\xi^{k'}, .) >=< \alpha_k, \xi^{k'} > \Omega_k - \alpha_k < \Omega_k, \xi^{k'} > . \tag{5.3.6}$$

But $<\Omega_k, \xi^{k'}>= 0$, $\forall k, k'$. Therefore, equality (5.3.6) becomes

$$< d\Omega_k, (\xi^{k'}, .) >=< \alpha_k, \xi^{k'} > \Omega_k.$$

Now

$$\Omega_k = \sum_{i=1}^{m} \lambda_{ik} \sum_{j=1}^{n} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) da^i_j.$$

Performing the exterior derivative, we get

$$d\Omega_k = \sum_{i,j,s,l} \lambda_{ik} \left(\frac{\partial^2 C^i}{\partial a^s_l \partial a^i_j} - \frac{\partial x^j}{\partial a^s_l} \right) da^s_l \wedge da^i_j + \sum_{i,j,s,l} \frac{\partial \lambda_{ik}}{\partial a^s_l} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) da^s_l \wedge da^i_j$$

Using (5.1.5), we can write $d\Omega_k$ as

$$d\Omega_k = \sum_{i,j,s,l} \left(\frac{\partial \lambda_{ik}}{\partial a^s_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^s_l} \right) da^s_l \wedge da^i_j$$

Now, we apply the 2-form $d\Omega_k$ to the vector field $\xi^{k'}$, we find that

$$\begin{aligned} < d\Omega_k, (\xi^{k'}, .) > &= \sum_{i,j,l} \left(\frac{\partial \lambda_{ik}}{\partial a^{k'}_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^{k'}_l} \right) a^{k'}_l da^i_j \\ &- \sum_{j,s,l} \left(\frac{\partial \lambda_{k'k}}{\partial a^s_l} \sum_r \frac{\partial x^r}{\partial a^{k'}_j} a^{k'}_r - \lambda_{k'k} \frac{\partial x^j}{\partial a^s_l} \right) a^{k'}_j da^s_l \\ &= \sum_{i,j,l} \frac{\partial \lambda_{ik}}{\partial a^{k'}_l} a^{k'}_l \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r da^i_j + \lambda_{k'k} \sum_{j,s,l} \frac{\partial x^j}{\partial a^s_l} a^{k'}_j da^s_l. \end{aligned}$$

Then, using Lemma (5.6) we have

$$\sum_l \frac{\partial \lambda_{ik}}{\partial a^{k'}_l} a^{k'}_l \quad = \quad \lambda_{ik}(\delta^{k'}_k - \delta^{k'}_i).$$

$$= \delta^{k'}_k \sum_i \lambda_{ik} \sum_{r,j} \frac{\partial x^r}{\partial a^i_j} a^i_r da^i_j + \lambda_{k'k} \left(\sum_{j,l} \frac{\partial x^j}{\partial a^{k'}_l} a^{k'}_j da^{k'}_l - \sum_{j,r} \frac{\partial x^r}{\partial a^{k'}_j} a^{k'}_r da^{k'}_j \right).$$

We end up with

$$< d\Omega_k, (\xi^{k'}, .) >= \delta^{k'}_k \sum_i \lambda_{ik} \sum_{r,j} \frac{\partial x^r}{\partial a^i_j} a^i_r da^i_j = \delta^{k'}_k \Omega_k.$$

Depending on the above formulas, we get

$$< d\Omega_k, (\xi^{k'}, .) >=< \alpha_k, \xi^{k'} > \sum_i \lambda_{ik} \sum_{r,j} \frac{\partial x^r}{\partial a^i_j} a^i_r da^i_j = \delta^{k'}_k \sum_i \lambda_{ik} \sum_{r,j} \frac{\partial x^r}{\partial a^i_j} a^i_r da^i_j.$$

We conclude that the differential 1-form α_k must satisfy $< \alpha_k, \xi^{k'} >= \delta^{k'}_k$ by seeting $\alpha_k = \sum_{s,l} R^l_{ks}(A) da^s_l$.

2. Now equation (5.3.5) can be written as

$$\sum_{i,j,s,l} \left(\frac{\partial \lambda_{ik}}{\partial a^s_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^s_l} \right) da^s_l \wedge da^i_j = \sum_{i,j,s,l} \left(R^l_{ks}(A) \lambda_{ik} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r \right) da^s_l \wedge da^i_j$$

$$\sum_{i,j,s,l} \left(\frac{\partial \lambda_{ik}}{\partial a^s_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^s_l} - R^l_{ks}(A) \lambda_{ik} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r \right) da^s_l \wedge da^i_j = 0. \quad (5.3.7)$$

Write the previous equation as

$$\sum_{i,j,s,l} (\Gamma_k)^{sl}_{ij} da^s_l \wedge da^i_j = 0,$$

where

$$(\Gamma_k)^{sl}_{ij} = \frac{\partial \lambda_{ik}}{\partial a^s_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^s_l} - R^l_{ks}(A) \lambda_{ik} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r.$$

Then equation (5.3.7) is satisfied if and only if $(\Gamma_k)^{sl}_{ij} = (\Gamma_k)^{ij}_{sl}$ for any given $k \in \{1, ..., m\}$ and all $1 \leq i, s \leq m$ and $1 \leq j, l \leq n$. So, we get the erquired symmetry conditions.

■

Corollary 5.12.1. *Suppose that Conditions (5.3.4) are satisfied. Then*

(a) $S_i = S_i^T$, *for all* $i = 1, ..., m$.
where S_i^{jl} *is the* $n \times n$ *matrix whose* jl*-entry is given by*

$$S_i^{jl} = \frac{\partial x^j}{\partial a^i_l} - R^j_{ii}(A) \sum_r \frac{\partial x^r}{\partial a^i_l} a^i_r$$

(b) $\frac{\partial x^l}{\partial a^i_j} + R^j_{ki}(A) \sum_r \frac{\partial x^r}{\partial a^k_l} a^k_r - \tau_{ik} \sum_{j'} \frac{\partial x^l}{\partial a^i_{j'}} a^k_{j'} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r = \lambda_{ik} \left(\frac{\partial x^j}{\partial a^k_l} - \tau_{ik} \sum_{j'} \frac{\partial x^{j'}}{\partial a^k_l} a^k_{j'} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r \right),$

where $\tau_{ik} = (\sum_{r,j'} \frac{\partial x^r}{\partial a^i_{j'}} a^i_r a^k_{j'})^{-1} = ((a^i)^T (D_{a^i} x^r) a^k)^{-1}$.

Proof. If $s = i = k$ then, using the fact that $\lambda_{ii} = 1$, relations (5.3.4) boil down to the following symmetry conditions

$$\frac{\partial x^j}{\partial a^i_l} - R^j_{ii}(A) \sum_r \frac{\partial x^r}{\partial a^i_l} a^i_r = \frac{\partial x^l}{\partial a^i_j} - R^l_{ii}(A) \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r.$$

So we get (a). To prove (b), it suffices to take $s = k$ and $i \neq k$ in (5.3.4) which writes down in this case as

$$\frac{\partial \lambda_{ik}}{\partial a^k_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^k_l} - R^l_{kk}(A) \lambda_{ik} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r = -\frac{\partial x^l}{\partial a^i_j} - R^j_{ki}(A) \sum_r \frac{\partial x^r}{\partial a^k_l} a^k_r. \quad (5.3.8)$$

Now, multiply both sides of the last equality by a^k_j, summing over j,

$$\frac{\partial \lambda_{ik}}{\partial a^k_l} \sum_{r,j'} \frac{\partial x^r}{\partial a^i_{j'}} a^i_r a^k_{j'} - \lambda_{ik} \sum_{j'} \frac{\partial x^{j'}}{\partial a^k_l} a^k_{j'} - R^l_{kk}(A) \lambda_{ik} \sum_{r,j'} \frac{\partial x^r}{\partial a^i_{j'}} a^i_r a^k_{j'}$$

$$= -\sum_{j'} \frac{\partial x^l}{\partial a^i_{j'}} a^k_{j'} - \sum_{j'} R^{j'}_{ki}(A) a^k_{j'} \sum_r \frac{\partial x^r}{\partial a^k_l} a^k_r.$$

But,

$$\sum_{j'} R^{j'}_{ki}(A) a^k_{j'} = 0.$$

Therefore, we have

$$\frac{\partial \lambda_{ik}}{\partial a^k_l} \sum_{r,j'} \frac{\partial x^r}{\partial a^i_{j'}} a^i_r a^k_{j'} - \lambda_{ik} \sum_{j'} \frac{\partial x^{j'}}{\partial a^k_l} a^k_{j'} - R^l_{kk}(A) \lambda_{ik} \sum_{r,j'} \frac{\partial x^r}{\partial a^i_{j'}} a^i_r a^k_{j'} = -\sum_{j'} \frac{\partial x^l}{\partial a^i_{j'}} a^k_{j'}.$$

Solving to get the following formula

$$\frac{\partial \lambda_{ik}}{\partial a^k_l} = \lambda_{ik} \tau_{ik} \sum_{j'} \frac{\partial x^{j'}}{\partial a^k_l} a^k_{j'} + \lambda_{ik} R^l_{kk}(A) - \tau_{ik} \sum_{j'} \frac{\partial x^l}{\partial a^i_{j'}} a^k_{j'}. \quad (5.3.9)$$

Substitute back into (5.3.8), we get

$$\lambda_{ik}\tau_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i+\lambda_{ik}R_{kk}^l(A)\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i-\tau_{ik}\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i$$

$$-\lambda_{ik}\frac{\partial x^j}{\partial a_l^k}-R_{kk}^l(A)\lambda_{ik}\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i=-\frac{\partial x^l}{\partial a_j^i}-R_{ki}^j(A)\sum_r\frac{\partial x^r}{\partial a_l^k}a_r^k.$$

Then,

$$\lambda_{ik}\tau_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i-\tau_{ik}\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i-\lambda_{ik}\frac{\partial x^j}{\partial a_l^k}$$

$$=-\frac{\partial x^l}{\partial a_j^i}-R_{ki}^j(A)\sum_r\frac{\partial x^r}{\partial a_l^k}a_r^k.$$

Rearranging, this condition can be written as

$$\frac{\partial x^l}{\partial a_j^i}+R_{ki}^j(A)\sum_r\frac{\partial x^r}{\partial a_l^k}a_r^k-\tau_{ik}\sum_{j'}\frac{\partial x^l}{\partial a_{j'}^i}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i$$

$$=\lambda_{ik}\left(\frac{\partial x^j}{\partial a_l^k}-\tau_{ik}\sum_{j'}\frac{\partial x^{j'}}{\partial a_l^k}a_{j'}^k\sum_r\frac{\partial x^r}{\partial a_j^i}a_r^i\right)$$

■

Remark 5.3.1. *We can use condition (b) to determine the functions* λ_{ik}.

5.4 Particular Case: $m = n = 2$

We consider a multi-constraint maximization problem of the form

$$\max_x f(x)$$

such that

$$C^1(A) = a_1^1 x^1 + a_2^1 x^2$$

$$C^2(A) = a_1^2 x^1 + a_2^2 x^2.$$

Define the Lagrangian function

$$L(x, \lambda) = f(x) + \lambda_1(C^1(A) - (a_1^1 x^1 + a_2^1 x^2)) + \lambda_2(C^2(A) - (a_1^2 x^1 + a_2^2 x^2))$$

with $x \in \mathbb{R}^2_{++}$, and $\lambda \in \mathbb{R}^2_{++}$. We take the derivative with respect to the control variables x, and the first order conditions for interior maximum are

$$\frac{\partial f}{\partial x^j} = \sum_{k=1}^{2} \lambda_k a_j^k, \qquad j = 1, 2.$$

Define the value function of this problem by

$$V(A) = \max_x \{ f(x) + \lambda_1(C^1(A) - (a_1^1 x^1 + a_2^1 x^2))$$
$$+ \lambda_2(C^2(A) - (a_1^2 x^1 + a_2^2 x^2)) \}.$$

Differentiating the function $V(A)$ with respect to a_j^i, we get

$$\frac{\partial V}{\partial a_j^i} = \sum_{k=1}^{2} \lambda_k \frac{\partial C^k}{\partial a_j^i} - \lambda_i x^j.$$

But, $D_{a^i} C^k = 0$ if $i \neq k$. Then,

$$\frac{\partial V}{\partial a_j^i} = \lambda_i \frac{\partial C^i}{\partial a_j^i} - \lambda_i x^j.$$

Define a family of differential 1-forms ω^1, ω^2 by

$$\omega^i = \sum_{j=1}^{2} \left(\frac{\partial C^i}{\partial a^i_j} - x^j \right) da^i_j.$$

Differentiating the function ω^i with respect to a^k_l

$$d\omega^i = \sum_{j,l}^{2} \frac{\partial^2 C^i}{\partial a^i_l \partial a^i_j} da^i_l \wedge da^i_j - \sum_{j,k,l}^{2} \frac{\partial x^j}{\partial a^k_l} da^k_l \wedge da^i_j.$$

The coefficients in the first summation are symmetric, so we end up with

$$d\omega^i = - \sum_{j,k,l}^{2} \frac{\partial x^j}{\partial a^k_l} da^k_l \wedge da^i_j.$$

The i^{th} constraint is $C^i(A) = \sum_{j=1}^{2} a^i_j x^j = a^i_1 x^1 + a^i_2 x^2$. Differentiating both sides of this equality with respect to a^i_j, we obtain

$$\frac{\partial C^i}{\partial a^i_j} = \sum_{r=1}^{2} \frac{\partial x^r}{\partial a^i_j} a^i_r + x^j$$

and rearranging the above formula, we get

$$\frac{\partial C^i}{\partial a^i_j} - x^j = \sum_{r=1}^{2} \frac{\partial x^r}{\partial a^i_j} a^i_r \ .$$

Thus,

$$\omega^i = \sum_{r,j=1}^{2} \frac{\partial x^r}{\partial a^i_j} a^i_r da^i_j \ .$$

The necessary and sufficient conditions for mathematical integration in this case given by:

Theorem 5.13. *Given the family of 1-forms Ω_1, Ω_2. Then*

$$\Omega_k \wedge d\Omega_k = 0$$

if and only if

1. *there exist a set of rank one $n \times m$ matrices R_1, R_2. that satisfy the conditions $R_k(a^i)^T = \delta^i_k$.*

2. *for all $1 \leq i, s \leq 2$, $1 \leq j, l \leq 2$ the following conditions are satisfied.*

$$\frac{\partial \lambda_{ik}}{\partial a^s_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^s_l} - R^l_{ks}(A) \lambda_{ik} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r$$
$$= \frac{\partial \lambda_{sk}}{\partial a^i_j} \sum_r \frac{\partial x^r}{\partial a^s_l} a^s_r - \lambda_{sk} \frac{\partial x^l}{\partial a^i_j} - R^j_{ki}(A) \lambda_{sk} \sum_r \frac{\partial x^r}{\partial a^s_l} a^s_r. \tag{5.4.1}$$

5.5 Economic Integration

In this section, we solve the economic integration problem. We start by giving the following theorem.

Theorem 5.14. *Given a function $x(A) \in \mathbb{R}^n_{++}$ be a zero$-$homogeneous in a^i, $i = 1, ..., m$ and a family of strictly positive functions λ_{ik} , $1 \leq i, k \leq m$ all of class C^2 that satisfies homogeneity conditions (5.6) defined in a neighbourhood $\mathcal{V}$ of some point $\bar{A}$ such that $\lambda_{ti}\lambda_{sk} = \lambda_{si}\lambda_{tk}$ for all λ_{ik}, $1 \leq i, k, s, t \leq m$. Then, there exist $m+1$ functions $\mu_i, ..., \mu_k$ and V, defined in a possibly smaller neighbourhood $\mathcal{U} \subset \mathcal{V}$, such that $\mu_k dV = \Omega_k$ if and only if conditions (5.3.4) are satisfied in $\mathcal{V}$.*

Proof. Given the functions $x(A)$ and λ_{ik}, $1 \leq i, k \leq m$ as in the statement of the theorem. Define a family of 1-forms Ω_k, $k = 1, ..., m$ as

$$\Omega_k = \sum_{s=1}^{m} \lambda_{sk} \omega^s,$$

where ω^s is the 1-form defined by

$$\omega^s = \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a_j^s} a_r^s da_j^s.$$

Now, $\Omega_k \wedge d\Omega_k = 0$ if and only if there exist two functions μ_k and V_k such that

$$\Omega_k = \mu_k dV_k.$$

But by Lemma(5.7)

$$\Omega_i \wedge \Omega_k = \mu_i \mu_k dV_i \wedge dV_k = 0.$$

Therefore, $dV_k = \phi_{ik}(A)dV_i$, $\forall i, k = 1, ..., m$ for some function ϕ_{ik}. So we can set $dV_1 = ... = dV_m = dV$. ■

We have the following result.

Lemma 5.15. *Suppose that $V(A)$ is the value function, $x(A)$ is a solution and $\lambda(A)$ is the associated vector of Lagrange multipliers for problem (P),*

$$C(A) = Ax(A).$$

Then we have

$$D^2_{a^i}V(A) = \lambda_i(A)(D^2_{a^i}C^i(a^i) - D_{a^i}x(A)) + D_{a^i}\lambda_i(A)(D_{a^i}C^i(a^i) - x)^T. \quad (5.5.1)$$

Moreover, the $n \times n$ matrix $D^2_{a^i}C^i(a^i) - D_{a^i}x(A)$ is symmetric and positive semi-definite on $\{D_{a^i}V\}^\perp$.

Proof.

$$D_{a^i}V = \lambda_i(D_{a^i}C^i(a^i) - x). \quad (5.5.2)$$

Differentiating equation (5.5.2), and we get the first order conditions

$$D^2_{a^i}V(A) = \lambda_i(A)(D^2_{a^i}C^i(a^i) - D_{a^i}x(A)) + D_{a^i}\lambda(A)(D_{a^i}C^i(a^i) - x)^T.$$

and the positively semi-definite result follows from the fact the value function V is quasi-convex and the $C^i(a^i)$ are convex with respect to a^i, and symmetric by summation of two symmetric matrix.

■

Lemma 5.16. *Let $x(A)$ be a solution of problem $(\mathcal{P})$ and $C(A) = Ax(A)$. Then*

$$\sum_r \frac{\partial^2 x^r}{\partial a_l^k \partial a_j^i} a_r^s + \frac{\partial x^l}{\partial a_j^i} \delta_k^s + \frac{\partial x^j}{\partial a_l^k} \delta_s^i = \frac{\partial^2 C^s}{\partial a_l^k \partial a_j^i} \delta_s^i \delta_s^k. \tag{5.5.3}$$

Moreover, if $C^i(a^i)$ is a convex function then the $n \times n$ matrix $\mathbf{M}^i$ where

$$\mathbf{M}_{jl}^i = \sum_r \frac{\partial^2 x^r}{\partial a_l^i \partial a_j^i} a_r^i + \frac{\partial x^l}{\partial a_j^i}_k + \frac{\partial x^j}{\partial a_l^i}$$

is symmetric and positive semi-definite.

Proof. The s^{th} constraint takes the form

$$(a^s)^T x(A) = C^s(a^s).$$

Differentiating both sides of this equality with respect to a_j^i,

$$\frac{\partial C^s(a^s)}{\partial a_j^i} \delta_s^i = \sum_{r=1}^n \frac{\partial x^r}{\partial a_j^i} a_r^s + x^j \delta_s^i. \tag{5.5.4}$$

Differentiating the equation (5.5.4) with respect to a_l^k,

$$\sum_r \frac{\partial^2 x^r}{\partial a_l^k \partial a_j^i} a_r^s + \frac{\partial x^l}{\partial a_j^i} \delta_k^s + \frac{\partial x^j}{\partial a_l^k} \delta_s^i = \frac{\partial^2 C^s}{\partial a_l^k \partial a_j^i} \delta_s^i \delta_s^k.$$

Thus, we have equation (5.5.3). Positivity follows from the convexity of $C^i(a^i)$ by theorem 2.4, and symmetric by summation of two symmetric matrix. ∎

Lemma 5.17. *Let $x(A)$ and $C(A) = Ax(A)$. Then the matrix T^i defined by*

$$T_{jl}^i = \sum_r \frac{\partial^2 x^r}{\partial a_l^i \partial a_j^i} a_r^i + \frac{\partial x^l}{\partial a_j^i}.$$

is symmetric and positive semi−definite on the subspace $\{(a^i)^T D_{a^i} x\}^\perp$.

Proof. From the above equation (5.5.3) we have $T^i + D_{a^i} x = D_{a^i}^2 C^i$. Using this equation and the result (5.1.3), we get

$$D_{a^i}^2 V = \lambda_i T^i + \frac{1}{\lambda_i} (D_{a^i} \lambda_i)(D_{a^i} V)^T.$$

The result follows from the last equality, the quasi-convexity of V with respect to a^i and the result that

$$D_{a^i}V = \lambda_i((a^i)^T D_{a^i}x).$$

■

The following theorem solves the economic integration problem.

Theorem 5.18. *Let $x(A) \in \mathbb{R}^n_{++}$, $\lambda_{ik}(A) > 0$ be given functions defined on a neighbourhood $\mathcal{U}$ of some point $\bar{A} \in \mathbb{R}^{mn}_{++}$ where x is zero$-$homogeneous in a^i, and λ_{ik} satisfy homogeneity conditions (5.6). Define $C(A) = Ax(A)$. Suppose that the following conditions are satisfied in $\mathcal{U}$ for all $i, k = 1, ..., m$.*

(a) $\lambda_{ti}\lambda_{sk} = \lambda_{si}\lambda_{tk}$ for all $1 \leq i, k, s, t \leq m$.

(b)

$$\begin{aligned}&\frac{\partial \lambda_{ik}}{\partial a^s_l}\sum_r \frac{\partial x^r}{\partial a^i_j}a^i_r - \lambda_{ik}\frac{\partial x^j}{\partial a^s_l} - R^l_{ks}(A)\lambda_{ik}\sum_r \frac{\partial x^r}{\partial a^i_j}a^i_r \\ = &\frac{\partial \lambda_{sk}}{\partial a^i_j}\sum_r \frac{\partial x^r}{\partial a^s_l}a^s_r - \lambda_{sk}\frac{\partial x^l}{\partial a^i_j} - R^j_{ki}(A)\lambda_{sk}\sum_r \frac{\partial x^r}{\partial a^s_l}a^s_r.\end{aligned}$$

(c) The matrix $\mathbf{M}^i$ is positive semi-definite.

(d) The restriction of the matrix T^i to $\{(a^i)^T D_{a^i}x\}^\perp$ is positive semi$-$definite.

Then, there exist positive functions $\lambda_1, ..., \lambda_m$ and a function V which is quasi-convex with respect to a^i for each i, defined in a neighbourhood $\mathcal{V} \subset \mathcal{U}$ such that

$$D_{a^i}V = \lambda_i(D_{a^i}C^i - x).$$

Proof. The condition (c) implies that the function $C^i(a^i)$ is convex. Consider the family of 1-forms $\Omega_1, ..., \Omega_m$ defined by

$$\begin{aligned}\Omega_k &= \sum_{i,j}\lambda_{ik}\left(\frac{\partial C^i}{\partial a^i_j} - x^j\right)da^i_j. \\ &= \sum_{i=1}^m \lambda_{ik}\sum_{r,j=1}^n \frac{\partial x^r}{\partial a^i_j}a^i_r da^i_j.\end{aligned}$$

Conditions (b) are equivalent to $\Omega_k \wedge d\Omega_k = 0$. Using Darboux theorem, the last equation is satisfied if and only if there exist two functions μ_k and V, where μ is $1-$homogeneous, and V is a zero$-$homogeneous such that $\mu_k dV = \Omega_k$. Note that V is independent of k. Therefore, we have

$$\mu_k \frac{\partial V}{\partial a^i_j} = \lambda_{ik} \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a^i_j} \ a^i_r. \tag{5.5.5}$$

Now, multiply both sides of the last equation (5.5.5) by a^k_j, and adding up

$$\mu_k \sum_j \frac{\partial V}{\partial a^i_j} a^k_j = \lambda_{ik} \sum_{r,j=1}^{n} \frac{\partial x^r}{\partial a^i_j} a^i_r \ a^k_j.$$

Using $\tau_{ik} = (\sum_{r,j'} \frac{\partial x^r}{\partial a^i_{j'}} a^i_r a^k_{j'})^{-1}$, where τ_{ik} is homogeneous of degree -1 in a^i, homogeneous of degree 0 in a^k, and homogeneous of degree -1 in $a^{k'}$, $\forall \ k' \neq i, k$.
We can write

$$\mu_k = \frac{\lambda_{ik}}{\tau_{ik}} \frac{1}{(a^k)^T D_{a^i} V} \quad , \quad i \neq k.$$

It follows that

$$\tau_{ik} \ \mu_k \ (a^k)^T D_{a^i} V = \lambda_{ik}(A) \ > \ 0 \tag{5.5.6}$$

for all A in sufficiently small neighbourhood of some point $\bar{A}$. Substitute for λ_{ik} in (5.5.5), we get

$$\mu_k dV = \sum_i \mu_k \tau_{ik}((a^k)^T D_{a^i} V)\left(\frac{\partial C^i}{\partial a^i_j} - x^j\right) da^i_j.$$

Canceling μ_k of the last equation, and setting

$$\tau_{ik}((a^k)^T D_{a^i} V) = \frac{\sum_j \frac{\partial V}{\partial a^i_j} a^k_j}{\sum_{r,j} \frac{\partial x^r}{\partial a^i_j} a^k_r a^i_r}.$$

Define a family of functions λ_i, $1 \leq i \leq m$ by

$$\lambda_i = \tau_{ik}((a^k)^T D_{a^i} V), \quad for \ some \quad k \neq i.$$

It follows from (5.5.6) that $\lambda_i > 0$ in a neighbourhood of $\bar{A}$.
Then

$$dV = \sum_{i=1}^{m} \lambda_i \left(\frac{\partial C^i}{\partial a^i_j} - x^j\right) da^i_j.$$

It remains to prove that the function V has the required positivity conditions. Note that

$$\frac{\partial^2 V}{\partial a_l^s \partial a_j^i} = \sum_{i=1}^{m} \lambda_i \left(\sum_{r=1}^{n} \frac{\partial^2 x^r}{\partial a_l^s \partial a_j^i} a_r^i + \frac{\partial x^l}{\partial a_j^i \delta_s^i} \right) + \frac{\partial \lambda_i}{\partial a_l^s} \sum_{r=1}^{n} \frac{\partial x^r}{\partial a_j^i} a_r^i.$$

Using relations (5.5.3), we can write $D_A^2 V$ as

$$\frac{\partial^2 V}{\partial a_l^s \partial a_j^s} = \lambda_s T_{jl}^s + \frac{\partial \lambda_s}{\partial a_l^s} \sum_{r=1}^{n} \frac{\partial x^r}{\partial a_j^s} a_r^s. \tag{5.5.7}$$

Take a vector $\varrho \in \{D_{a^s} V\}^{\perp}$; that is, ϱ satisfies the condition

$$\sum_{j=1}^{n} \sum_{r=1}^{n} \frac{\partial x^r}{\partial a_j^s} a_r^s \varrho_j = 0.$$

It follows that

$$\sum_{j,l=1}^{n} \frac{\partial^2 V}{\partial a_l^s \partial a_j^s} \varrho_j \varrho_l = \lambda_s \sum_{j,l=1}^{n} T_{jl}^s \varrho_j \varrho_l \geq 0.$$

We conclude that the matrix $D_{a^s}^2 V$ is positive semi$-$definite on $\{D_{a^s} V\}^{\perp}$; that is, V is quasi-convex with respect to a^s. ■

5.6 Duality

After solving the mathematical problem, we get functions $\lambda_1, ...\lambda_m$ and V that have the required properties. The question now is how to get a concave (or quasi-concave) objective function.

In the single constraint case, if $V(a)$ is strongly convex (meaning that the Hessian is positive), then $f(x) = \min_a \{V(a)|a'x \le c(a)\}$ is quasi-convex. The objective function can be obtained from the value function using the duality relation

$$f(x) = \min\{V(A)|(a^i)^T x(A) = C^i(A)\}.$$

The function f is not necessarily quasi-concave.

However, we can introduce a class of functions that is stable under duality. We need to define the following space

$$\mathcal{E}(A) = \{v = (v^1, ..., v^m) \in \mathbb{R}^{mn} | (v^i)^T D_{a^i} V = 0,\ i = 1, ..., m\}.$$

We now recall the definitions of QE-convex and QE-concave.

Definition 31. *Let $\mathcal{U} \subset \mathbb{R}^n_{++}$ and $\mathcal{V} \subset \mathbb{R}^{mn}_{++}$. Suppose that $C(A)$ is a convex mapping. Then,*

1. *We say that a function $f(x)$ is locally QE-concave if*

$$\forall x^* \in \mathcal{U}, \exists A^* \in \mathcal{V} \text{ such that } f(x^*) = \max_{x \in \mathcal{U}} \{f(x)|A^* x = C(A^*)\}.$$

2. *We say that a function $V(A)$ is locally QE-convex if*

$$\forall A^* \in \mathcal{V}, \exists x^* \in \mathcal{U} \text{ such that } V(A^*) = \min_{A \in \mathcal{V}} \{V(A)|Ax^* = C(A)\}.$$

We have the following theorems.

Theorem 5.19. *The value function $V(A)$ is locally QE-convex if $D^2_A V$ is positive definite on $\mathcal{E}(A)$.*

Proof. Let $\mathcal{V}$ be a neighbourhood of a point $\bar{A}$ in which the function V is defined. The assumption that $D_p^2 V$ is positive definite on $\mathcal{E}(A)$ for all $A \in \mathcal{V}$ implies that if $v = (v^1, ..., v^m) \in \mathcal{E}$ such that $(a^1 + v^1, ..., a^m + v^m) \in \mathcal{V}$ then

$$V(a^1 + v^1, ..., a^m + v^m) > V(a^1, ..., a^m). \tag{5.6.1}$$

To show that V is locally QE-convex, suppose that A^* is given. Let x^* be such that

$$V(A^*) = \min_A \{V(A) | Ax^* = C(A)\}.$$

Take

$$x^*(A) = D_{a^i} C^i(a^{i*}) - \frac{1}{\lambda_i(A^*)} D_{a^i} V(A^*)$$

and

$$\lambda_i(A^*) = \tau_{ik}(A^*)((a^{k*})^T D_{a^i} V(A^*)),$$

where $\tau_{ik}^{-1} = ((a^i)^T (D_{a^i} x^r) a^k)$.

The point A^* satisfies the first order optimality conditions. Its clear that $A^* x^*(A^*) = C(a^{i*})$. The point A^* satisfies the second order condition for minimum which is the positive definiteness of $D_A^2 V$ on $\mathcal{E}(A^*)$. ■

Now, we need to show that the function

$$f(x) = \min_{A \in \mathcal{V}} \{V(A) | Ax = C(A)\}$$

is locally QE-concave if V is locally QE-convex. Let $f(x)$ be a given locally QE-concave fuction. Define a function $V : \mathcal{V} \subset \mathbb{R}^{mn}_{++} \to \mathbb{R}$ by

$$V(A) = \max_{x \in \mathcal{U}} \{f(x) | Ax = C(A)\}$$

Define also the function $f^*(x) = \min_{A \in \mathcal{V}} \{V(A) | Ax = C(A)\}$.

Suppose that the function $V(A)$ is defined in a neighbourhood of some point $\bar{A} \in \mathbb{R}^{mn}_{++}$, then $\mathcal{U} = \{x \in \mathbb{R}^n_{++} | Ax = C(A),\ \forall A \in \mathcal{V}\}$.
The following theorem establishes duality between f and V.

Theorem 5.20. *[1] If V is locally QE-convex then f^* is locally QE-concave. Moreover, $f^* = f$ throughout $\mathcal{U}$ if f is locally QE-concave.*

Proof. Let $x^* \in \mathcal{U}$ such that $A^* \in \arg min\{V(A)|Ax^* = C(A),\ A \in \mathcal{V}\}$. Such A^* exists because V is locally QE-convex. We want to show that

$$x^* \in \arg max\{f^*(x)|A^*x = C(A^*), x \in \mathcal{U}\}.$$

The inequality $f^* \leq V(A^*) = f^*(x^*)$ implies that f^* attains its maximum under the constraints $A^*x = C(A^*)$ at x^* we conclude that f^* is locally QE-concave. Now, we show that $f^* = f$ on $\mathcal{U}$. let $x^* \in \mathcal{U}$ such that there exists A^* such that $x^* \in \arg max\{f^*(x)|A^*x = C(A^*), x \in \mathcal{U}\}$. Such x^* exists because f is locally QE-concave. Therefore, $V(A) \geq V(A^*) = f(x^*)$ for all $A \in \mathcal{V}$ such that $Ax^* = C(A)$. This means that V attains its minimum under $Ax^* = C(A)$ at A^*, from which, by definition, $f^*(x^*) = V(A^*) = f(x^*)$. This implies $f^*(x^*) = f(x^*)$, because x^* is an arbitrary point in $\mathcal{U}$, we conclude that $\mathcal{U}^*(x) = \mathcal{U}(x)$, $\forall x \in \mathcal{U}$. We have shown at the same time, that if A^* is the solution or belongs to the solution set of

$$\min_A\{V(A)|Ax = C(A),\ A \in \mathcal{V}\},$$

then $x^*(A^*)$ is the solution, or belong to the solution set of

$$\max_x\{f^*(x)|A^*x = C(A^*),\ x \in \mathcal{U}\},$$

and conversely. ∎

Equation (5.5.7) implies that, on the space $\mathcal{E}(A)$, we have for any fixed $k_0 \in \{1, ..., m\}$:

$$\frac{1}{\lambda_{k_0}}\frac{\partial^2 V}{\partial a_l^k \partial a_j^i} = \lambda_{ik_0} T_{jl}^i + \frac{1}{\lambda_{k_0}}\frac{\partial \lambda_i}{\partial a_l^k}\sum_r \frac{\partial x^r}{\partial a_j^i} a_r^i = \mathbf{K}_{jl}^{ik}.$$

Clearly, the assumption of positive definiteness of D^2V on the subspace $\mathcal{E}(A)$ can now be stated in terms of observable functions, namely λ_{ik_0} and x. Moreover, it is a stronger condition than the assumption of positive definiteness of T^i on $\{(a^i)^T D_{a^i} x\}^\perp$ as required in theorem . To put all pieces of the puzzle together, we state the following theorem that gives the solution of the inverse problem

Theorem 5.21. *Let $x(A) \in \mathbb{R}^n_{++}$, $\lambda_{ik}(A) > 0$ be given functions defined on a neighbourhood $\mathcal{U}$ of some point $\bar{A} \in \mathbb{R}^{mn}_{++}$. Define $C(A) = Ax(A)$. Suppose that the following conditions are satisfied in $\mathcal{U}$ for all $i, k = 1, ..., m$.*

(a) $\lambda_{ti}\lambda_{sk} = \lambda_{si}\lambda_{tk}$ *for all* $1 \leq i, k, s, t \leq m$.

(b)

$$\frac{\partial \lambda_{ik}}{\partial a^s_l} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r - \lambda_{ik} \frac{\partial x^j}{\partial a^s_l} - R^l_{ks}(A)\lambda_{ik} \sum_r \frac{\partial x^r}{\partial a^i_j} a^i_r$$
$$= \frac{\partial \lambda_{sk}}{\partial a^i_j} \sum_r \frac{\partial x^r}{\partial a^s_l} a^s_r - \lambda_{sk} \frac{\partial x^l}{\partial a^i_j} - R^j_{ki}(A)\lambda_{sk} \sum_r \frac{\partial x^r}{\partial a^s_l} a^s_r.$$

(c) The matrix $\mathbf{M}^i$ *is positive semi-definite.*

(d) The restriction of the tensor $\mathbf{K}^{ik}_{jl}$ *to the subspace* $\mathcal{E}(A)$ *is positive definite.*

Then, there exists a locally QE-concave function $f(x)$ *such that*

$$x(A) \in arg\max\{f(x)|Ax = C(A)\}.$$

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